

# Lipschitz Algebras and Derivations

## II. Exterior Differentiation

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Basic aspects of differential geometry can be extended to various non-classical settings: Lipschitz manifolds, rectifiable sets, sub-Riemannian manifolds, Banach

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derivative, we give a general construction which applies to any metric space equipped with a  $\sigma$ -finite measure and produces the desired result in all of the above cases. It agrees with Cheeger's construction when the latter is defined. It also applies to an important class of Dirichlet spaces, where, however, the known first-order differential calculus in general differs from ours (although the two are related).

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## 1. INTRODUCTION

This paper is a continuation of [78]. There we considered derivations of  $L^\infty(X, \mu)$  into a certain kind of bimodule and constructed an associated metric on  $X$ . Here we take a metric space  $M$  as given and consider only derivations into modules whose left and right actions coincide (monomodules). This allows the extraction of a kind of differentiable structure on  $M$ . The exterior derivative is the universal derivation, into a certain type of bimodule, which is compatible with the metric in a special sense. The main results of this paper were stated (without proof) in Section 7.3 of [79].

Surprisingly, this construction requires no serious conditions on  $M$ ; in particular, it need not be a manifold. But actually several lines of research point in this direction. First, in Connes' noncommutative geometry [16, 17] it appears that however one makes sense of the noncommutative version of the property of being a manifold, it is basically unrelated to the

noncommutative version of differentiable structure (and the latter is, from a functional-analytic point of view, simpler). This algebraic approach to differentiable structure is already explicit in [66, 67]. Second, Gromov [33, 34] (see also [11]) has gotten mileage out of the fact that relatively sophisticated geometric notions, such as sectional curvature, can be treated in a purely metric fashion. Also along these lines, there is significant current interest in using Poincaré inequalities to pursue some kind of differential analysis on exotic metric spaces (see [14, 36, 39, 40, 47, 64, 65]). Third, Sauvageot was able to construct an exterior derivative from initial data consisting only of a Dirichlet form with certain properties [62, 63], and others have also treated Dirichlet spaces geometrically [6, 69–72]. Finally, recent work on fractals [43, 44, 49, 61] has a strong geometric flavor, again indicating that manifold structure is not necessary for some kind of differential analysis.

Our main result is the identification of a module of measurable 1-forms and a first-order exterior derivative on any metric space equipped with a  $\sigma$ -finite Borel measure. The consequences are availability of basic differential geometric tools in non-classical settings and unification of previous work along these lines.

Actually, the measure itself is not relevant to our construction, only its measure class. In most finite-dimensional examples Hausdorff measure (see, e.g., [29]) provides the appropriate measure class. To be precise, there is at most one value of  $p \in [0, \infty)$  such that  $p$ -dimensional Hausdorff measure is  $\sigma$ -finite but nonzero, and this is the desired measure. So in a restricted setting one can regard our construction as proceeding from the metric alone. However, we will want to consider some infinite-dimensional examples (meaning that  $p$ -dimensional Hausdorff measure is infinite for all  $p$ ), and here it appears that the measure must be added as an independent ingredient. There may also be occasional finite-dimensional examples where one wants to use something other than Hausdorff measure class (e.g., see the comment following Theorem 41 and the second example considered in Theorem 58).

It is interesting to note that for much of what we do the infinite-dimensional Riemannian/Hilbert module case is just as tractable as the general finite-dimensional case. (See Corollary 24 and the comment following Theorem 10.)

### A. *Non-classical Spaces*

By the “classical” setting we mean the case of a smooth (or at worst  $C^1$ ) Riemannian manifold. Applying our construction in this case, using no structure besides the distance function and the Lebesgue measure class, produces the usual (measurable) differential geometric data associated with the manifold.

More generally, let  $M$  be a Lipschitz manifold. By this we mean that  $M$  is a metric space which is locally bi-Lipschitz equivalent to the unit ball in  $\mathbf{R}^n$ ; see [21, 51, 60] for background. These spaces still possess measurable 1-forms and a first-order exterior derivative, and our construction still recovers them.

In a different direction, one can generalize Riemannian manifolds by sub-Riemannian (or “Carnot–Carathéodory”) spaces [5]. Here the metric is defined in terms of paths which are tangent to a fixed subbundle of the tangent bundle, and our construction then recovers this subbundle. (Cf. the more sophisticated view of tangent spaces for left-invariant sub-Riemannian metrics on Lie groups described in [4, 56].)

Any sub-Riemannian manifold may be viewed as a Gromov–Hausdorff limit of Riemannian manifolds (e.g., see [34]); but in general these limits need not themselves be Riemannian, sub-Riemannian, or even manifolds. Nonetheless, Peter Petersen has pointed out to me that one can define a measure on any such limit by treating Gromov–Hausdorff convergence as Hausdorff convergence within a larger space (see [58]) and taking a weak\* limit of measures on the approximating spaces. So our construction will also apply in this situation, but I have not pursued this.

Rectifiable sets in the sense of geometric measure theory [26, 54] constitute another broad class, similar to the Lipschitz manifolds, in which a geometric approach has proved valuable. Here too there is a natural tangent space defined almost everywhere which can be recovered from the metric alone.

Generalizing to infinite dimensions, one can treat metric spaces that are locally bi-Lipschitz equivalent to the unit ball of a Banach space and are equipped with a measure that satisfies a mild condition. Examples of such spaces are path spaces of the type considered in [24, 55], for example. (But not spaces of continuous paths; see the discussion of Wiener space below.)

Non-rectifiable fractals—specifically, fractals with non-integral Hausdorff dimension—have also been investigated in various ways that have some geometric flavor [41, 43–45, 49, 73]. This is seen most explicitly in the renormalized Laplace operator and the Gauss–Green formula of [43]. There has also been interesting work on diffusion processes in this setting [2, 3, 46, 61]. Unfortunately, our construction is generally vacuous for sets of this type, so it seems that we have nothing to contribute here. I believe there is no meaningful “renormalized” tangent bundle for these sets.

But the diffusion processes just mentioned lack an important property: the associated Dirichlet forms do not admit a *carré du champ* (see [9]). In the presence of this extra hypothesis there is a natural underlying metric and, independently, an elegant construction of an exterior derivative [62]. But the latter in general is different from our exterior derivative. Their relationship is addressed in Proposition 57.

As a special case, one may consider the standard diffusion process and associated Dirichlet form on Wiener space. This does admit a *carré du champ*, and the two exterior derivatives agree. This example has been considered in detail and can be described in an elementary fashion, without reference to Dirichlet forms (see [9, 75]). It is important to realize that although Wiener space is a Banach manifold, this is not really relevant to its differentiable structure: the Gross–Sobolev derivative is not directly derived from the metric which provides local Banach space structure. In particular, the tangent spaces carry inner products despite the fact that Wiener space, in its usual formulation, is not locally isomorphic to a Hilbert space.

Finally, there has been much recent interest in metric spaces which admit a Poincaré inequality, and exotic examples of such spaces have been constructed in [10, 36, 47]. The case that such spaces are differentiable in some sense is made in [39, 40, 64, 65], and an exterior derivative in this setting was defined in [14] concurrently with this paper (see Section 5.F).

We must also mention recent progress which has been made in several non-classical directions in geometry which do not clearly fit with the point of view taken here. One of these is Harrison’s work on nonsmooth chains [37], which deals with  $p$ -forms on nonsmooth sets. This is really a generalization in a different direction and possibly could be combined with our approach. Davies’ analysis on graphs [19, 20], based on Connes’ two-point space [17, Example VI.3.1], involves a notion of 1-form, but this is a discrete, not a differential, object. (The definition of  $p$ -forms for the product of a two-point space by an ordinary manifold is treated nicely in [48].) Finally, noncommutative geometry in the sense of Connes [16, 17] has become a major industry, with applications in many directions. It is unclear to what extent the idea of vector fields as derivations, which is central to this paper, is helpful in the noncommutative setting. The main reason for thinking not is that if  $A$  is a noncommutative algebra, then the set of derivations from  $A$  into itself generally is not a module over  $A$ . Still, some progress in this direction has been made (see [22, 53]).

## B. Plan of the Paper and Acknowledgments

We begin with some background material on the type of  $L^\infty(X)$ -modules of interest to us. These “abelian  $W^*$ -modules” are introduced and their structure analyzed in Section 2. Then in Section 3 we describe the notion of a “measurable metric” and its relation to derivations into bimodules. These two sections set up the fundamental notions that are needed in the sequel.

The two succeeding sections are the core of the paper. Our construction and its general properties are given in Section 4. Section 5 is devoted to examples; here we show that our construction produces the expected result

in most if not all of the cases where there is one, and we also consider a few new cases. Finally, in Section 6 we carefully consider the case of Dirichlet spaces.

The original source of motivation for this work was Sauvageot's pair of papers [62, 63], which show that one can have a meaningful exterior derivative in the absence of anything resembling manifold structure. It has also been my good fortune to have been given in person a great deal of information and advice on the topics considered here. In this regard Daniel Allcock, Al Baernstein, Jeff Cheeger, Renato Feres, Ken Goodearl, Jura Heinonen, Gary Jensen, Mohan Kumar, Michel Lapidus, Paul Muhly, Peter Petersen, Jürgen Schweizer, Stephen Semmes, Mitch Taibleson, and Ed Wilson all contributed to the mathematical content of this paper, and it is a pleasure to acknowledge their help.

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## 2. ABELIAN $W^*$ -MODULES

### A. Definitions

The scalar field will be real throughout. We will invoke several facts from the literature which involve complex scalars, but this raises no serious issues. In every case a trivial complexification argument justifies the application.

Let  $K$  be a compact Hausdorff space and let  $E$  be a module over  $C(K)$ . Recall that  $E$  is a *Banach module* if it is also a Banach space and its norm satisfies  $\|f\phi\| \leq \|f\| \|\phi\|$  for all  $f \in C(K)$  and  $\phi \in E$ .

Following [23], we say that  $E$  is a  $C(K)$ -normed module if there exists a map  $|\cdot|: E \rightarrow C(K)$  such that  $|\phi| \geq 0$ ,  $\|\phi\| = \|\phi\|_\infty$ , and  $|f\phi| = |f| |\phi|$  for all  $f \in C(K)$  and  $\phi \in E$ . This map is to be thought of as a fiberwise norm; according to [59, Corollary 6], it is unique if it exists. By [23, pp. 48 and 59, Proposition 2] we have the following equivalent characterization of  $C(K)$ -normed modules:

**THEOREM 1.** *Let  $E$  be a Banach module over  $C(K)$ . Then  $E$  is a  $C(K)$ -normed module if and only if it is isometrically isomorphic to the set of continuous sections of some (F) Banach bundle over  $K$ .*

Here, as in [23], (F) Banach bundles are Fell bundles  $B$  of Banach spaces  $B_x$ , i.e.  $B = \bigcup_{x \in K} B_x$ . Their defining property is that the Banach space norm is continuous as a map from  $B$  to  $\mathbf{R}$ .

Now let  $X = (X, \mu)$  be a measure space. We will assume throughout that  $X$  is  $\sigma$ -finite, although everything we do should work in the more general case that  $X$  is finitely decomposable. By this we mean that  $X$  can be expressed as a (possibly uncountable) disjoint union of finite measure subsets,  $X = \bigcup X_i$ , such that  $A \subset X$  is measurable if and only if  $A \cap X_i$  is measurable for all  $i$ , in which case  $\mu(A) = \sum \mu(A \cap X_i)$ . (The spaces  $L^\infty(X)$  for  $X$  finitely decomposable are precisely the real parts of abelian von Neumann algebras.)

Now  $L^\infty(X)$  is isomorphic to  $C(K)$  for some extremely disconnected compact Hausdorff space  $K$ , so the notion of a  $C(K)$ -normed module specializes to this case. In this situation we have a further equivalence [59, Theorem 9]:

**THEOREM 2.** *Let  $E$  be a Banach module over  $L^\infty(X)$ . The following are equivalent:*

- (a)  *$E$  is an  $L^\infty(X)$ -normed module;*
- (b) *there is a compact Hausdorff space  $K$ , an isometric algebra homomorphism  $\iota$  from  $L^\infty(X)$  into  $C(K)$ , and an isometric linear map  $\pi$  from  $E$  into  $C(K)$  such that  $\pi(f\phi) = \iota(f)\pi(\phi)$  for all  $f \in L^\infty(X)$  and  $\phi \in E$ ;*
- (c)  *$\|\phi\| = \max(\|p\phi\|, \|(1-p)\phi\|)$  for all  $\phi \in E$  and any projection  $p \in L^\infty(X)$ .*

Theorem 2b is an abelian version of the notion of an operator module (see [25]). Modules with this property were used heavily in [78] and are central to this paper as well. The precise class of modules which we need is specified in the next definition.

**DEFINITION 3.** A Banach module  $E$  over  $L^\infty(X)$  is an *abelian  $W^*$ -module* if it satisfies the equivalent conditions of Theorem 2, and in addition is a dual Banach space such that the product map  $L^\infty(X) \times E \rightarrow E$  is separately weak\*-continuous in each variable.

The notion of duality is central to our analysis of abelian  $W^*$ -modules. This concept is given in the next definition; following it, we give a module version of the Hahn–Banach theorem.

**DEFINITION 4.** Let  $E$  be a Banach module over  $C(K)$ . Then its *dual module*  $E'$  is the set of norm-bounded  $C(K)$ -module homomorphisms from  $E$  into  $C(K)$ . Give each such homomorphism its norm as an operator between Banach spaces, and define the module operation by

$$(f\Phi)(\phi) = f \cdot \Phi(\phi)$$

for  $f \in C(K)$ ,  $\phi \in E$ , and  $\Phi \in E'$ . It is easy to check that  $E'$  is a Banach module over  $C(K)$ .

**THEOREM 5.** *Let  $E$  be an  $L^\infty(X)$ -normed module, let  $E_0 \subset E$  be a submodule, and let  $\Phi_0: E_0 \rightarrow L^\infty(X)$  be a norm-one module homomorphism. Then there is a norm-one module homomorphism  $\Phi: E \rightarrow L^\infty(X)$  such that  $\Phi|_{E_0} = \Phi_0$ .*

*Proof.* By Zorn's lemma, it will suffice to consider the case that there exists  $\phi_0 \in E$  such that  $E$  consists of the set of elements of the form  $f\phi_0 + \psi$  with  $f \in L^\infty(X)$  and  $\psi \in E_0$ .

Observe that  $|\Phi_0(\psi)| \leq |\psi|$  almost everywhere, for any  $\psi \in E_0$ . For if not, there would be a positive measure subset  $A \subset X$  such that  $|\Phi_0(\psi)| \geq |\psi| + \varepsilon$  almost everywhere on  $A$ . But then

$$\|\Phi_0(\chi_A \psi)\| = \|\Phi_0(\chi_A \psi)\| = \|\chi_A |\Phi_0(\psi)|\| \geq \|\chi_A \psi\| + \varepsilon,$$

contradicting the fact that  $\|\Phi_0\| = 1$ .

The remaining argument is a simple reworking of the usual proof of the Hahn–Banach theorem. Namely, observe that if  $f_1, f_2 \in L^\infty(X)$ ,  $f_1, f_2 \geq 0$ , and  $\psi_1, \psi_2 \in E_0$  then

$$\Phi_0(f_2 \psi_1 - f_1 \psi_2) \leq |f_2 \psi_1 - f_1 \psi_2| \leq f_1 |f_2 \phi_0 + \psi_2| + f_2 |f_1 \phi_0 + \psi_1|.$$

From this it follows that

$$(-|f_2 \phi_0 + \psi_2| - \Phi_0(\psi_2))/f_2 \leq (|f_1 \phi_0 + \psi_1| - \Phi_0(\psi_1))/f_1$$

on the common support of  $f_1$  and  $f_2$ . Also, both sides lie in the interval  $[-\|\phi_0\|, \|\phi_0\|]$  wherever they are defined. Taking the supremum of the left side in  $L^\infty(X)$  (assigning the default value  $-\|\phi_0\|$  whenever  $f_2 = 0$ ), we obtain  $g \in L^\infty(X)$  such that

$$(-|f\phi_0 + \psi| - \Phi_0(\psi))/f \leq g \leq (|f\phi_0 + \psi| - \Phi_0(\psi))/f$$

holds on the support of  $f$ , for all  $f \in L^\infty(X)$  with  $f \geq 0$ . By decomposition of an arbitrary  $f \in L^\infty(X)$  into its positive and negative parts, this implies that

$$|fg + \Phi_0(\psi)| \leq |f\phi_0 + \psi|; \quad (*)$$

the inequality is immediate where  $f$  is positive, has already been proven where  $f = 0$ , and holds in the negative case by replacing  $\psi$  with  $-\psi$ . So defining  $\Phi(f\phi_0 + \psi) = fg + \Phi_0(\psi)$  for all  $f \in L^\infty(X)$  and  $\psi \in E_0$  produces a norm-one module homomorphism  $\Phi$  which extends  $\Phi_0$ .

( $\Phi$  is well-defined by the following argument. Suppose  $f_1\phi_0 + \psi_1 = f_2\phi_0 + \psi_2$ . Then  $(f_1 - f_2)\phi_0 + (\psi_1 - \psi_2) = 0$ , so  $(*)$  implies that  $|(f_1 - f_2)g + \Phi_0(\psi_1 - \psi_2)| = 0$ . Thus

$$(f_1g + \Phi_0(\psi_1)) - (f_2g + \Phi_0(\psi_2)) = (f_1 - f_2)g + \Phi_0(\psi_1 - \psi_2) = 0,$$

which shows that  $\Phi$  is well-defined.) ■

**COROLLARY 6.** *If  $E$  is an  $L^\infty(X)$ -normed module then for any  $\phi \in E$  there exists  $\Phi \in E'$  with  $\|\Phi\| = 1$  and  $\Phi(\phi) = |\phi|$ . The natural map from  $E$  into  $E''$  is isometric.*

*Proof.* The natural map is automatically nonexpansive. Conversely, given any  $\phi \in E$  define  $\Phi_0(f\phi) = f|\phi|$ ; this is a norm-one module homomorphism defined on  $E_0 = \{f\phi: f \in L^\infty(X)\}$ , so it extends to a norm-one  $\Phi \in E'$  by the theorem. Since  $\|\Phi(\phi)\| = \||\phi|\| = \|\phi\|$ , we are done. ■

## B. Characterizations

First, we characterize abelian  $W^*$ -modules in terms of duality.

**THEOREM 7.** *If  $E$  is a Banach module over  $L^\infty(X)$  then  $E'$  is an abelian  $W^*$ -module. Conversely, any abelian  $W^*$ -module over  $L^\infty(X)$  is isometrically and weak\*-continuously isomorphic to the dual of some  $L^\infty(X)$ -normed module.*

*Proof.* To show that  $E'$  is an  $L^\infty(X)$ -normed module, we check the property given in Theorem 2c. Let  $p \in L^\infty(X)$  be a projection and let  $\Phi \in E'$ . Then  $\|p\Phi\| \leq \|p\| \|\Phi\| = \|\Phi\|$ , and similarly  $\|(1-p)\Phi\| \leq \|\Phi\|$ , so

$$\|\Phi\| \geq \max(\|p\Phi\|, \|(1-p)\Phi\|).$$

Conversely, given  $\varepsilon > 0$  find  $\phi \in E$  such that  $\|\phi\| = 1$  and  $\|\Phi(\phi)\| \geq \|\Phi\| - \varepsilon$ . Then

$$\|\Phi(\phi)\| = \max(\|p\Phi(\phi)\|, \|(1-p)\Phi(\phi)\|),$$

so we must have

$$\begin{aligned} \max(\|p\Phi\|, \|(1-p)\Phi\|) &\geq \max(\|p\Phi(\phi)\|, \|(1-p)\Phi(\phi)\|) \\ &= \|\Phi(\phi)\| \geq \|\Phi\| - \varepsilon. \end{aligned}$$

This verifies the  $L^\infty(X)$ -normed property.

To see that  $E'$  is a dual Banach space, let  $Y$  be the set-theoretic cartesian product of the unit ball of  $E$  with the unit ball of  $L^1(X)$ , and define



$T: E' \rightarrow l^\infty(Y)$  by  $(T\Phi)(\phi, f) = \int \Phi(\phi) f$ . It is straightforward to check that  $T$  is isometric. If  $(\Phi_i)$  is a bounded universal net in  $E'$  then  $\Phi(\phi) = \lim \Phi_i(\phi)$  (where the limit is taken in the weak\* topology on  $L^\infty(X)$ ) defines  $\Phi \in E'$  such that  $T\Phi_i \rightarrow T\Phi$  pointwise. This shows that the unit ball of  $T(E')$  is weak\*-closed in  $l^\infty(Y)$ ; hence  $T(E')$  is weak\*-closed by the Krein–Smulian theorem [18, Theorem 12.1] and therefore a dual space. Since  $T$  is isometric,  $E'$  is also a dual space, and on bounded sets its weak\* topology satisfies  $\Phi_i \rightarrow \Phi$  if  $\Phi_i(\phi) \rightarrow \Phi(\phi)$  weak\* in  $L^\infty(X)$  for all  $\phi \in E$ .

Finally, if  $f_i$  is a bounded net in  $L^\infty(X)$  which converges weak\* to  $f$  then for any  $\phi \in E$  and  $\Phi \in E'$  we have

$$(f_i \Phi)(\phi) = f_i \cdot \Phi(\phi) \rightarrow f \cdot \Phi(\phi) = (f\Phi)(\phi),$$

so the map  $(f, \Phi) \mapsto f\Phi$  is continuous in the first variable, and if  $\Phi_i$  is a bounded net in  $E'$  which converges weak\* to  $\Phi$ , then for any  $f \in L^\infty(X)$  and  $\phi \in E$  we have

$$(f\Phi_i)(\phi) = f \cdot \Phi_i(\phi) \rightarrow f \cdot \Phi(\phi) = (f\Phi)(\phi),$$

so the map  $(f, \Phi) \mapsto f\Phi$  is also continuous in the second variable. (It is enough to check continuity on bounded nets by the Krein–Smulian theorem.) So  $E'$  is an abelian  $W^*$ -module, completing the proof of the first statement.

For the converse, let  $E$  be an abelian  $W^*$ -module. By [25, Theorem 4.1] there exist a complex Hilbert space  $H$ , an isometric and weak\*-continuous map  $\iota$  of  $E$  onto a weak\*-closed subspace of  $B(H)_{\text{sa}}$ , and an isometric weak\*-continuous algebra homomorphism  $\pi$  from  $L^\infty(X)$  into  $B(H)_{\text{sa}}$ , such that

$$\iota(f) \pi(\phi) = \pi(f\phi) = \pi(\phi) \iota(f)$$

for all  $f \in L^\infty(X)$  and  $\phi \in E$ . Here  $B(H)_{\text{sa}}$  denotes the set of bounded self-adjoint operators on  $H$ .

Now let  $\phi \in E$  and  $\varepsilon > 0$ ; we will find  $\Phi \in E'$  which is weak\*-continuous and satisfies  $\|\Phi\| = 1$  and  $\|\Phi(\phi)\| \geq \|\phi\| - \varepsilon$ . By the preceding we may assume  $L^\infty(X) \subset B(H)_{\text{sa}}$ ,  $E \subset B(H)_{\text{sa}}$ , and  $f\phi = \phi f$  for all  $f \in L^\infty(X)$  and  $\phi \in E$ .

Since  $\phi \in B(H)$  is self-adjoint, we can find a unit vector  $v \in H$  such that  $|\langle \phi v, v \rangle| \geq \|\phi\| - \varepsilon$ . Let  $X_0 \subset X$  be the largest subset such that  $\langle f v, v \rangle = 0$  for no  $f \in L^\infty(X_0)$  besides  $f = 0$ , i.e.  $X_0$  is the support of the vector state given by  $v$ . By a standard argument (e.g. see [42]), there is then a unique map—a conditional expectation— $\Phi$  from  $E$  into  $L^\infty(X_0) \subset L^\infty(X)$  with the property that

$$\langle \Phi(\psi) f v, v \rangle = \langle \psi f v, v \rangle$$

for all  $f \in L^\infty(X)$  and  $\psi \in E$ , and it is straightforward to verify that  $\Phi \in E'$  has the desired properties. Thus, letting  $E_0$  be the weak\*-continuous part of  $E'$ , it follows that the natural map from  $E$  onto  $E'_0$  is noncontractive. But it is automatically nonexpansive and weak\*-continuous, so  $E \cong E'_0$ . ■

We now deduce a measurable version of Theorem 2b.

**COROLLARY 8.** *There exist a finitely decomposable measure space  $Y$ , an isometric weak\*-continuous algebra homomorphism  $\iota$  from  $L^\infty(X)$  into  $L^\infty(Y)$ , and an isometric weak\*-continuous linear map  $\pi$  from  $E$  into  $L^\infty(Y)$ , such that  $\iota(f)\pi(\phi) = \pi(f\phi)$  for all  $f \in L^\infty(X)$  and  $\phi \in E$ .*

*Proof.* Retain the notation used in the proof of Theorem 7. For each  $\Phi \in E_0$  with  $\|\Phi\| = 1$ , let  $X_\Phi$  be a copy of  $X$ ; then let  $Y = \bigcup X_\Phi$  be their disjoint union. Let  $\iota$  be the diagonal embedding of  $L^\infty(X)$  into  $L^\infty(Y)$ , and define  $\pi: E \rightarrow L^\infty(Y)$  by  $\pi(\phi) = \Phi(\phi)$  on  $X_\Phi$ .

It follows from Theorem 7 that  $\pi$  is isometric and weak\*-continuous, and the remainder is easy. ■

Since  $L^\infty(X)$  is semihereditary [32], every finitely-generated projective module over  $L^\infty(X)$  is isomorphic to a direct sum of ideals (e.g., see [13]). (I am indebted to Ken Goodearl for this argument.) Finitely-generated abelian  $W^*$ -modules enjoy a similar characterization.

**LEMMA 9.** *For each  $n \in \mathbf{N}$ , if there is any subset  $X_n \subset X$  with the property that  $\chi_{X_n}E$  is generated by  $n$  elements, then there is a maximal such subset.*

*Proof.* Let  $\Gamma$  be the collection of all measurable subsets  $S$  of  $X$  with the property that  $\chi_S E$  is generated by  $n$  elements. (By “generated” we mean that  $\chi_S E$  is the smallest abelian  $W^*$ -module which contains the  $n$  generators.) It is clear that if  $S \in \Gamma$  and  $T \subset S$  then  $T \in \Gamma$ . Also, if  $\{S_k\} \subset \Gamma$  and the  $S_k$  are disjoint, then let  $\phi_1^k, \dots, \phi_n^k$  be a generating set for  $\chi_{S_k} E$  for each  $k$ . Normalizing, we may suppose that  $\|\phi_i^k\| \leq 1$  for all  $i$  and  $k$ . As the  $S_k$  are disjoint, we may then take the weak\* sum  $\phi_i = \sum_k \phi_i^k$  ( $1 \leq i \leq n$ ), and the  $\phi_i$  then generate  $\chi_S E$  where  $S = \bigcup S_k$ .

Thus,  $\Gamma$  is closed under subsets and disjoint unions; it follows that it contains a maximal element (up to null sets). ■

**THEOREM 10.** *Let  $E$  be a finitely generated abelian  $W^*$ -module over  $L^\infty(X)$ . Then there is a partition of  $X$ ,  $X = \bigcup_{n=1}^m X_n$ , and for each  $x \in X_n$  a norm  $\|\cdot\|_x$  on  $\mathbf{R}^n$ , such that  $E$  is isometrically and weak\*-continuously isomorphic to the set of bounded measurable functions  $f$  such that  $f|_{X_n}$  takes  $X_n$  into  $\mathbf{R}^n$  for  $1 \leq n \leq m$ , with norm given by  $\|f\| = \text{esssup } \|f(x)\|_x$ .*

*Proof.* The differences of the sets described in Lemma 9 provide the desired partition of  $X$ . By Theorem 2c, it suffices to restrict attention to one block of the partition; thus we may suppose that  $E$  is generated by  $n$  elements  $\phi_1, \dots, \phi_n$  and for any positive measure subset  $A \subset X$ ,  $\chi_A E$  is not generated by fewer than  $n$  elements.

For  $x \in X$  and  $a = (a_1, \dots, a_n) \in \mathbf{R}^n$ , define

$$\|a\|_x = |a_1 \phi_1 + \dots + a_n \phi_n|(x).$$

This is a seminorm for almost every  $x$  by [59, Proposition 2], and the module of bounded measurable sections of the trivial bundle with fiber  $\mathbf{R}^n$ , equipped with this family of seminorms, is isometric to a weak\*-dense subset of  $E$  via the identification of  $f: X \rightarrow \mathbf{R}^n$  with  $\sum (\pi_i \circ f) \phi_i$ , where  $\pi_i$  is the  $i$ th coordinate projection on  $\mathbf{R}^n$ . Furthermore, the weak\*-topology of  $E$  agrees with the weak\*-topology of  $L^\infty(X)$  on the  $L^\infty(X)$ -span of each  $\phi_i$ , hence the identification just described is a weak\*-homeomorphism; hence the module of sections is isometric to all of  $E$ , since its unit ball is already weak\* compact.

Finally, suppose  $\|\cdot\|_x$  fails to be a norm on a set  $A$  of positive measure. For each  $x \in A$  let  $B_x = \{a \in \mathbf{R}^n : \|a\|_x = 0\}$ ; then  $B = \bigcup B_x$  is an (F) Banach bundle over  $L^\infty(A)$  and so there exists a bounded, nonzero, measurable section  $f: A \rightarrow \mathbf{R}^n$  with  $f(x) \in B_x$  for all  $x$  by [27, Appendix]. Let  $\phi = \sum (\pi_i \circ f) \phi_i$ , setting  $\phi = 0$  off of  $A$ ; without loss of generality we may assume  $|\pi_1 \circ f| \geq \varepsilon > 0$  on a positive measure subset  $A_0$  of  $A$ . Then  $\phi_1$  is expressible as a linear combination of  $\phi$  together with  $\phi_2, \dots, \phi_n$ . But  $|\phi|(x) = 0$  for almost every  $x \in A$ , hence  $\phi = 0$ , so that  $\phi_2, \dots, \phi_n$  generate  $\chi_{A_0} E$ . This contradicts the reduction made in the first paragraph of the proof, so we conclude that  $\|\cdot\|_x$  is a norm for almost every  $x \in X$ . ■

Theorem 10 can be viewed as saying that any finitely generated abelian  $W^*$ -module is isometrically isomorphic to the module of bounded measurable sections of some bundle of finite-dimensional Banach spaces. An analogous result holds for non-finitely-generated Hilbert modules [57, Theorem 3.12; see also 74]; in this case the fibers are Hilbert spaces.

Morally, it should be true without the finitely generated assumption that every abelian  $W^*$ -module is isometric to the module of bounded measurable sections of some bundle of dual Banach spaces. However, measure-theoretic complications make it difficult to formulate a satisfying general result of this type.

We require one final construction: the tensor product of abelian  $W^*$ -modules. For our purposes the appropriate definition is the following.

**DEFINITION 11.** Let  $E = E'_0$  and  $F = F'_0$  be abelian  $W^*$ -modules over  $L^\infty(X)$ . We define  $E \otimes F = E \otimes_{L^\infty(X)} F$  to be the set of bounded module

maps from  $E_0$  into  $F$ , or equivalently the set of bounded module maps from  $F_0$  into  $E$ . It is straightforward to check that this is again an abelian  $W^*$ -module.

(A bounded module map  $T: E_0 \rightarrow F$  gives rise to a map from  $F_0 \subset F'$  to  $E$  by taking adjoints, and vice versa.)

### 3. MEASURABLE METRICS

#### A. Definitions

In some ways the pointwise aspect of metrics does not interact well with non-atomic measures. A helpful alternative, which involves only distances between positive measure sets, was formulated in [77] and also used in [78]. It is the following.

DEFINITION 12. Let  $M = (M, \mu)$  be a  $\sigma$ -finite measure space and let  $\Omega$  be the collection of all positive measure subsets of  $M$ , modulo null sets. A *measurable pseudometric* is a map  $\rho: \Omega^2 \rightarrow [0, \infty]$  such that

$$\begin{aligned}\rho(A, A) &= 0 \\ \rho(A, B) &= \rho(B, A) \\ \rho\left(\bigcup_{n=1}^{\infty} A_n, B\right) &= \inf_n \rho(A_n, B) \\ \rho(A, C) &\leq \sup_{B' \subset B} (\rho(A, B') + \rho(B', C))\end{aligned}$$

for all  $A, B, C, A_n \in \Omega$ .

We will need the notion of a measurable pseudometric in Section 6, but it is not really necessary elsewhere in the paper. So the reader who so wishes is, up to Section 6, free to assume that any measurable pseudometric under discussion is actually a pointwise metric. By this we mean that the distance between positive measure sets is given by

$$\rho(A, B) = \sup \inf \{ \rho_0(x, y) : x \in A', y \in B' \} = \sup \rho_0(A', B'),$$

where  $\rho_0(x, y)$  is a pointwise metric and the sup is taken over all measurable sets  $A'$  and  $B'$  which differ from  $A$  and  $B$  by null sets. (This does not entail an essential loss in generality; see Theorem 20.)

We limit ourselves to a brief sketch of relevant facts about measurable metrics. For more details the reader can consult [77]. Proofs of Proposition 14 and Theorem 20 will be published elsewhere.

DEFINITION 13. Let  $M = (M, \mu, \rho)$  be a  $\sigma$ -finite measure space with measurable pseudometric. The *essential range* of  $f \in L^\infty(M)$  is the set of  $a \in \mathbf{R}$  such that  $f^{-1}(U)$  has positive measure for every neighborhood  $U$  of  $a$  (equivalently, it is the spectrum of  $f$  in the real Banach algebra  $L^\infty(M)$ ); and we let  $\rho_f(A, B)$  denote the distance, in  $\mathbf{R}$ , between the essential ranges of  $f|_A$  and  $f|_B$ .

The *Lipschitz number*  $L(f)$  of  $f$  is

$$L(f) = \sup\{\rho_f(A, B)/\rho(A, B) : A, B \in \Omega \text{ and } \rho(A, B) > 0\}$$

and  $\text{Lip}(M)$  is the set of all  $f \in L^\infty(M)$  for which  $L(f)$  is finite. With the norm  $\|f\|_L = \max(L(f), \|f\|_\infty)$ ,  $\text{Lip}(M)$  is easily seen to be a Banach space as well as a ring.

We say that  $\rho$  is a *measurable metric* if  $\text{Lip}(M)$  is weak\*-dense in  $L^\infty(M)$ . We also have the following equivalent condition, which, in the case of an atomic measure, is equivalent to the condition  $\rho(x, y) = 0 \Rightarrow x = y$ .

PROPOSITION 14. *A measurable pseudometric  $\rho$  is a measurable metric if and only if the underlying measurable  $\sigma$ -algebra is generated (up to null sets) by the sets  $A \in \Omega$  with the property that*

$$A \cap B = \emptyset \quad \Rightarrow \quad \text{there exists } B' \subset B \text{ such that } \rho(A, B') > 0$$

*for all  $B \in \Omega$ .*

We record two more basic facts. The first follows from [78, Theorem 9] and the second is [76, Theorem B]. (The latter was proven in [76] only for pointwise metrics, but the proof in the measurable case is essentially identical.)

PROPOSITION 15.  *$\text{Lip}(M)$  is a dual space, and on its unit ball the weak\*-topology agrees with the restriction of the weak\*-topology on  $L^\infty(M)$ .*

THEOREM 16. *Let  $M$  be a measurable metric space and let  $\mathcal{A}$  be a weak\*-closed unital subalgebra of  $\text{Lip}(M)$ . Suppose that there exists  $k \geq 1$  such that for every  $A, B \subset M$  we have*

$$\rho_f(A, B) = \rho(A, B)$$

*for some  $f \in \mathcal{A}$  with  $L(f) \leq k$ . Then  $\mathcal{A} = \text{Lip}(M)$ .*

## B. Derivations

Measurable metrics are closely connected to a natural class of derivations. These derivations are described in the following definitions.

**DEFINITION 17.** Let  $E$  be a bimodule over  $L^\infty(X)$  (with possibly different left and right actions) which is also a dual Banach space. It is an *abelian  $W^*$ -bimodule* if there exist a finitely decomposable measure space  $Y$ , an isometric weak\*-continuous linear map  $\pi$  from  $E$  into  $L^\infty(Y)$ , and isometric weak\*-continuous algebra homomorphisms  $\iota_l$  and  $\iota_r$  from  $L^\infty(X)$  into  $L^\infty(Y)$  such that

$$\pi(f\phi g) = \iota_l(f) \pi(\phi) \iota_r(g)$$

for all  $f, g \in L^\infty(X)$  and  $\phi \in E$ .

**DEFINITION 18.** Let  $E$  be an abelian  $W^*$ -bimodule over  $L^\infty(X)$ . An (unbounded)  *$W^*$ -derivation* from  $L^\infty(X)$  into  $E$  is then a linear map  $\delta$  from a weak\*-dense, unital subalgebra of  $L^\infty(X)$  into  $E$  with the property that  $\delta(fg) = f\delta(g) + \delta(f)g$  and whose graph is a weak\*-closed subspace of  $L^\infty(X) \oplus E$ .

The following theorem is a slight reformulation of the main result of [78]. We will use it in Section 6.

**THEOREM 19.** Let  $M = (X, \rho)$  be a measurable metric space. Then there is a  $W^*$ -derivation  $\delta$  from  $L^\infty(X)$  into an abelian  $W^*$ -bimodule whose domain equals  $\text{Lip}(M)$  and such that  $L(f) = \|\delta(f)\|$  for all  $f \in \text{Lip}(M)$ .

Conversely, let  $\delta$  be any  $W^*$ -derivation from  $L^\infty(X)$  into an abelian  $W^*$ -bimodule. Then there is a measurable metric  $\rho$  on  $X$  such that the domain of  $\delta$  equals  $\text{Lip}(M)$  and  $L(f) = \|\delta(f)\|$  for all  $f \in \text{Lip}(M)$ , where  $M = (X, \rho)$ .

## C. Metric realization

As we mentioned earlier, any genuine metric  $\rho_0$  on  $M$  gives rise to a measurable metric, by setting

$$\rho_0(A, B) = \inf_{x \in A, y \in B} \rho_0(x, y)$$

and then letting  $\rho(A, B)$  be the supremum of  $\rho_0(A', B')$  as  $A'$  and  $B'$  range over all measurable sets which differ from  $A$  and  $B$  by null sets. If  $\mu$  is atomic, it is not hard to see that every measurable metric on  $M$  comes from a unique pointwise metric in this manner.

But in general not every measurable metric on  $M$  arises in this way, and in the second part of Theorem 19 the use of measurable metrics is necessary. However, if one is willing to modify the set  $X$  one can always get a genuine underlying metric. This is in keeping with the algebraic point of view which regards the algebra  $L^\infty(X)$  as primary and the measure space  $X$  as noncanonical and secondary. We now state precisely how an arbitrary measurable metric can be reduced to an ordinary metric; this incidentally sharpens the results of [77, 78].

**THEOREM 20.** *Let  $M = (M, \mu, \rho_M)$  be a  $\sigma$ -finite measurable metric space. Then there is a  $\sigma$ -finite measure space  $N = (N, \nu)$  with a complete pointwise metric  $\rho_N$  and an isometric isomorphism of  $L^\infty(M)$  onto  $L^\infty(N)$  which carries  $\text{Lip}(M)$  onto  $\text{Lip}(N)$  (= the bounded measurable Lipschitz functions on  $N$ ) in a manner which preserves Lipschitz number.*

Note that in general there may be Lipschitz functions on  $N$  which are not measurable. There is no requirement that  $\rho_N$  be compatible with  $\nu$  in any sense, and in fact there are natural examples (for instance, any nonatomic space with  $\rho_N(x, y) = \infty$  whenever  $x \neq y$ ) where  $\nu$  is not a Borel measure with respect to the topology generated by  $\rho_N$ . This highlights the generality of the construction given in the next section.

## 4. 1-FORMS AND THE EXTERIOR DERIVATIVE

### A. The Construction

**DEFINITION 21.** Let  $M$  be a measurable metric space and let  $E$  be an abelian  $W^*$ -module over  $L^\infty(M)$ . A *metric derivation*  $\delta: \text{Lip}(M) \rightarrow E$  is a bounded weak\*-continuous linear map which satisfies the derivation identity  $\delta(fg) = f\delta(g) + \delta(f)g$  for all  $f, g \in \text{Lip}(M)$ .

The *module of measurable vector fields*  $\mathcal{X}(M)$  is the set of all metric derivations  $\delta: \text{Lip}(M) \rightarrow L^\infty(M)$ . The module action is given by  $(f \cdot \delta)(g) = f\delta(g)$  for  $f \in L^\infty(M)$ ,  $\delta \in \mathcal{X}(M)$ , and  $g \in \text{Lip}(M)$ . Using Theorem 2c, it is straightforward to check that  $\mathcal{X}(M)$  is an  $L^\infty(M)$ -normed module.

The *module of measurable 1-forms*  $\Omega(M)$  is the dual module of  $\mathcal{X}(M)$ ; that is,  $\Omega(M) = \mathcal{X}(M)'$ . By Theorem 7 this is an abelian  $W^*$ -module over  $L^\infty(M)$ .

In general  $\mathcal{X}(M)$  is not an abelian  $W^*$ -module over  $L^\infty(M)$ ; see Proposition 48a. This is true in the finitely generated and Hilbert module cases, however (Corollary 24). Also, the notion of metric derivation is strictly weaker than the notion of  $W^*$ -derivation given in Definition 18. In

particular, there may exist elements of  $\mathcal{X}(M)$  which are not  $W^*$ -derivations; see Proposition 48b.

By Theorem 10, if  $\Omega(M)$  is finitely generated then it can be realized as the module of bounded measurable sections of some bundle of finite-dimensional Banach spaces. The latter plays the role of the cotangent bundle. Again, something like this possibly involving infinite-dimensional dual Banach spaces should be true even when  $\Omega(M)$  is not finitely generated. These would act as cotangent spaces and their preduals as tangent spaces. In the finitely generated case we assuredly have a natural tangent bundle according to Corollary 24 and Theorem 10.

Additionally, in the Hilbert module case we have tangent and cotangent bundles, even if  $M$  is infinite-dimensional. Here they are bundles of Hilbert spaces, and the tangent and cotangent spaces at each point are naturally identified with each other (Corollary 24; see the comment following Theorem 10).

We should remark that Gromov has given a definition of the tangent space or “asymptotic cone” at a point of any metric space [34]. It appears to bear little relationship to our definition. For instance, the tangent space of Gromov at a boundary point of a manifold with boundary will be a half-space, whereas our tangent spaces are always Banach spaces. On the other hand, our tangent spaces will only be well defined almost everywhere.

By taking tensor products over  $L^\infty(M)$ , one can define the module of bounded measurable tensor fields of arbitrary type  $(r, s)$ . Together with the following definition this raises the question of whether higher order exterior derivatives must exist, i.e., whether  $d$  can in general be extended to the whole exterior algebra. I suspect the answer is no even in the finite-dimensional case, but this has been done for Lipschitz manifolds [31].

**DEFINITION 22.** The *exterior derivative* on  $M$  is the map  $d: \text{Lip}(M) \rightarrow \Omega(M)$  given by  $(df)(\phi) = \phi(f)$  for  $f \in \text{Lip}(M)$  and  $\phi \in \mathcal{X}(M)$ .

More generally, if  $E$  is any submodule of  $\mathcal{X}(M)$  there is a natural map  $d_E: \text{Lip}(M) \rightarrow E'$  given by the same formula,  $(d_E f)(\phi) = \phi(f)$ . Letting  $T: \Omega(M) \rightarrow E'$  be the natural projection, we have  $d_E = T \circ d$ .

**THEOREM 23.** *The exterior derivative  $d$  is a metric derivation. It is universal in the sense that if  $\delta: \text{Lip}(M) \rightarrow E$  is any metric derivation into an abelian  $W^*$ -module then there is a bounded weak\*-continuous  $L^\infty(M)$ -module map  $T: \Omega(M) \rightarrow E$  such that  $\delta = T \circ d$ . Furthermore,  $\|T\| = \|\delta\|$ .*

*Proof.* It is clear that  $d$  is linear. If  $f, g \in \text{Lip}(M)$  and  $\phi \in \mathcal{X}(M)$  then

$$d(fg)(\phi) = \phi(fg) = f\phi(g) + \phi(f)g = f(dg)(\phi) + (df)(\phi)g,$$



so  $d$  is a derivation. And if  $(f_i)$  is a bounded net in  $\text{Lip}(M)$  and  $f_i \rightarrow f$  weak\* then

$$(df_i)(\phi) = \phi(f_i) \rightarrow \phi(f) = (df)(\phi)$$

weak\* in  $L^\infty(M)$  for any  $\phi \in \mathcal{X}(M)$ , hence  $df_i \rightarrow df$  weak\* in  $\Omega(M)$ . This shows that  $d$  is a metric derivation.

Let  $\delta: \text{Lip}(M) \rightarrow E = E'_0$  be any metric derivation. (We may assume that  $E$  is a dual module by Theorem 7.) Define  $T_0: E_0 \rightarrow \mathcal{X}(M)$  by  $(T_0\phi)(f) = (\delta f)(\phi)$  for  $\phi \in E_0$  and  $f \in \text{Lip}(M)$ . Then  $T_0$  is a bounded  $L^\infty(M)$ -module map and it has a bounded, weak\*-continuous adjoint  $T: \Omega(M) \rightarrow E$ . For any  $f \in \text{Lip}(M)$  and  $\phi \in E$  we then have

$$T(df)(\phi) = (df)(T_0\phi) = (T_0\phi)(f) = (\delta f)(\phi),$$

so  $\delta = T \circ d$ . Also  $\|T\| = \|T_0\| = \|\delta\|$ . ■

**COROLLARY 24.** *If  $\Omega(M)$  is reflexive then  $\mathcal{X}(M)$  is an abelian  $W^*$ -module. In particular, this holds if  $\mathcal{X}(M)$  is finitely generated or if  $\mathcal{X}(M)$  satisfies the parallelogram law*

$$|\phi + \psi|^2 + |\phi - \psi|^2 = 2|\phi|^2 + 2|\psi|^2$$

(almost everywhere, for all  $\phi, \psi \in \mathcal{X}(M)$ ). In the latter case  $\mathcal{X}(M)$  and  $\Omega(M)$  are canonically isomorphic self-dual Hilbert modules.

*Proof.* Suppose  $\Omega(M)$  is reflexive. By this we mean that the natural map from  $\Omega(M)$  into  $\Omega(M)''$ —which is isometric by Corollary 6—is onto. This implies that every element of  $\Omega(M)'$  is weak\*-continuous. So for any  $\Phi \in \Omega(M)'$  the map  $f \mapsto \Phi(df)$  is a metric derivation from  $\text{Lip}(M)$  into  $L^\infty(M)$ ; i.e., it is an element of  $\mathcal{X}(M)$ . This shows that the natural map from  $\mathcal{X}(M)$  into  $\Omega(M)'$ —which, again, is isometric by Corollary 6—is onto, hence  $\mathcal{X}(M)$  is an abelian  $W^*$ -module by Theorem 7.

Suppose  $\mathcal{X}(M)$  is finitely generated. Then  $E = \mathcal{X}(M)''$  is a finitely generated abelian  $W^*$ -module, and Theorem 10 implies that  $E'$  is finitely generated as well (namely, it is the module of bounded measurable sections of the same vector bundle, equipped with the fiberwise dual norms). But  $\Omega(M)$  is a quotient of  $E' = \Omega(M)''$  by Theorem 5, so  $\Omega(M)$  is also finitely generated. By Theorem 10 we deduce that it is reflexive.

Now suppose  $\mathcal{X}(M)$  satisfies the parallelogram law. Then  $\mathcal{X}(M)$  is a Hilbert module by [59, Lemma 13] and so  $\Omega(M)$  is self-dual and hence reflexive by [57, Theorem 3.2]. From the first part of the corollary it now follows that  $\mathcal{X}(M) \cong \Omega(M)' \cong \Omega(M)$ .

In the finitely generated case we have a simple formula for the module of vector fields on a product space. I suspect it is false in general, but may be true if  $\mathcal{X}(M)$  and  $\mathcal{X}(N)$  are Hilbert modules.

**THEOREM 25.** *Let  $M$  and  $N$  be measurable metric spaces and suppose that  $\mathcal{X}(M)$  and  $\mathcal{X}(N)$  are finitely generated. Then*

$$\mathcal{X}(M \times N) \cong (\mathcal{X}(M) \otimes L^\infty(M \times N)) \oplus (\mathcal{X}(N) \otimes L^\infty(M \times N)),$$

where the first tensor product is taken over  $L^\infty(M)$  and the second is taken over  $L^\infty(N)$ , and  $\cong$  denotes isomorphism of  $L^\infty(M \times N)$ -modules.

*Proof.* Recall our version of tensor products of modules given in Definition 11. Here we are viewing  $L^\infty(M \times N)$  as an abelian  $W^*$ -module over either  $L^\infty(M)$  or  $L^\infty(N)$ .

We need not specify the product metric on  $M \times N$  exactly, since all natural choices are bi-Lipschitz equivalent. This ambiguity corresponds to a choice of the direct sum norm in the right side of the isomorphism. We do require that  $\rho(A \times B, A' \times B) = \rho(A, A')$  and  $\rho(A \times B, A \times B') = \rho(B, B')$  for all  $A, A' \subset M$  and  $B, B' \subset N$ . The easiest way to see that measurable metrics on  $M \times N$  satisfying these conditions exist is via Theorem 20.

We first define a map  $S$  from  $\mathcal{X}(M \times N)$  into the right side. To do this it suffices to separately define maps  $S_M: \mathcal{X}(M \times N) \rightarrow \mathcal{X}(M) \otimes L^\infty(M \times N)$  and  $S_N: \mathcal{X}(M \times N) \rightarrow \mathcal{X}(N) \otimes L^\infty(M \times N)$ . Since  $\Omega(M)$  is reflexive, Corollary 24 implies that  $\mathcal{X}(M) \cong \Omega(M)'$ , and so  $\mathcal{X}(M) \otimes L^\infty(M \times N)$  is identified with the set of bounded  $L^\infty(M)$ -module maps from  $\Omega(M)$  into  $L^\infty(M \times N)$ . Thus, to define  $S_M$  we must say how to produce such a map from an arbitrary derivation  $\delta \in \mathcal{X}(M \times N)$ . This is done by observing that  $\text{Lip}(M)$  naturally imbeds in  $\text{Lip}(M \times N)$ , hence  $\delta$  restricts to a metric derivation from  $\text{Lip}(M)$  into  $L^\infty(M \times N)$ , and the required module map is then given by the universality statement in Theorem 23.  $S_N$  is defined similarly. It is clear that  $\|S_M\|, \|S_N\| \leq 1$ .

Next, we show that  $S(\mathcal{X}(M \times N))$  contains  $\mathcal{X}(M) \otimes 1_{M \times N}$  and  $\mathcal{X}(N) \otimes 1_{M \times N}$ . Identifying these sets with  $\mathcal{X}(M)$  and  $\mathcal{X}(N)$  in the obvious way, we will define maps  $T_M: \mathcal{X}(M) \rightarrow \mathcal{X}(M \times N)$  and  $T_N: \mathcal{X}(N) \rightarrow \mathcal{X}(M \times N)$  such that  $S \circ T_M$  is the identity on  $\mathcal{X}(M)$  and  $S \circ T_N$  is the identity on  $\mathcal{X}(N)$ .

Let  $\delta \in \mathcal{X}(M)$  and let  $f \in \text{Lip}(M \times N)$ ; to define  $T_M(\delta)$  we must produce an element of  $L^\infty(M \times N)$ , or equivalently a bounded linear map from  $L^1(N)$  into  $L^\infty(M)$ . Letting  $g \in L^1(N)$ , the desired element of  $L^\infty(M)$  is then  $\delta(f_g)$ , where

$$f_g = \int f(x, y) g(y) dy$$

is a bounded Lipschitz function with  $L(f_g) \leq L(f) \|g\|_1$ . It is now a matter of unwrapping definitions to verify that  $S_M(T_M(\delta)) = \delta$  and  $S_N(T_M(\delta)) = 0$ . A similar argument proves the corresponding result for  $T_N$ .

Now  $\mathcal{X}(M) \otimes 1_{M \times N}$  and  $\mathcal{X}(N) \otimes 1_{M \otimes N}$  together algebraically generate the target space as an  $L^\infty(M \times N)$ -module. This can be seen by using the explicit structure of  $\mathcal{X}(M)$  and  $\mathcal{X}(N)$  given by Corollary 24 and Theorem 10, and it implies from the above that  $S$  is onto.  $S$  is also 1-1 since the algebraic tensor product of  $\text{Lip}(M)$  and  $\text{Lip}(N)$  is weak\*-dense in  $\text{Lip}(M \times N)$  by Theorem 16. So the open mapping theorem implies that  $S$  is an isomorphism. ■

We conclude this section with a technical criterion which is helpful in determining  $\mathcal{X}(M)$  in some examples.

**THEOREM 26.** *Let  $M$  be a measurable metric space. Let  $E$  be a submodule of  $\mathcal{X}(M)$  which is reflexive as an  $L^\infty(M)$ -module and suppose  $|df| = |d_E f|$  for all  $f$  in a weak\*-dense subspace of  $\text{Lip}(M)$ . Then  $E = \mathcal{X}(M)$ .*

*Proof.* Let  $S \subset \text{Lip}(M)$  be a weak\*-dense subspace such that  $|df| = |d_E f|$  for all  $f \in S$ . Let  $T: \Omega(M) \rightarrow E'$  be the restriction map, so that  $d_E = T \circ d$  and  $T$  is nonexpansive.

Moreover, we have  $|T(df)| = |d_E f| = |df|$  for all  $f \in S$ . We now claim that  $T$  remains isometric on the  $L^\infty(M)$ -span of these elements  $df$ . To see this let  $\sum_1^n f_i dg_i$  be a finite linear combination with  $f_i \in L^\infty(M)$  and  $g_i \in S$ , and let  $\varepsilon > 0$ . Without loss of generality suppose  $L(g_i) \leq 1$  for all  $i$ . Let  $A$  be a positive measure set on which

$$\left| \sum f_i dg_i \right| \geq \left\| \sum f_i dg_i \right\| - \varepsilon.$$

By shrinking  $A$ , we may assume that each  $f_i$  varies by at most  $\varepsilon/n$  on  $A$ . Choose  $a_i \in \mathbf{R}$  such that  $|f_i(x) - a_i| \leq \varepsilon/n$  for almost every  $x \in A$ .

Now

$$\left| \left( \sum f_i dg_i \right) - \left( \sum a_i dg_i \right) \right| \leq \sum |(f_i - a_i) dg_i| \leq \sum |f_i - a_i| L(g_i) \leq \varepsilon$$

on  $A$ , so

$$\left| d \left( \sum a_i g_i \right) \right| = \left| \sum a_i dg_i \right| \geq \left| \sum f_i dg_i \right| - \varepsilon \geq \left\| \sum f_i dg_i \right\| - 2\varepsilon$$

on  $A$ . Since  $|T(df)| = |df|$  for all  $f \in S$ , we then have

$$\left| \sum a_i d_E g_i \right| = \left| d_E \left( \sum a_i g_i \right) \right| = \left| d \left( \sum a_i g_i \right) \right| \geq \left\| \sum f_i dg_i \right\| - 2\varepsilon$$

on  $A$ . But finally  $|(\sum f_i d_E g_i) - (\sum a_i d_E g_i)| \leq \varepsilon$  on  $A$  by applying  $T$  to the inequality  $|(\sum f_i dg_i) - (\sum a_i dg_i)| \leq \varepsilon$ , and so we conclude that

$$\left\| \sum f_i d_E g_i \right\| \geq \left\| \sum f_i dg_i \right\| - 3\varepsilon.$$

Taking  $\varepsilon$  to zero completes the proof of the claim.

Let  $\phi \in \mathcal{X}(M)$ . Let  $E_0$  be the set of elements in  $E'$  of the form  $\sum f_i d_E g_i$  considered above, and let  $T^{-1}$  denote the isometric embedding of  $E_0$  into  $\Omega(M)$  given by  $T^{-1}(\sum f_i d_E g_i) = \sum f_i dg_i$ . Then the map  $\Phi \mapsto T^{-1}(\Phi)(\phi)$  is a bounded module homomorphism from  $E_0$  into  $L^\infty(M)$ , and by Theorem 5 this extends to an element of  $E'' = E$ . Thus there exists  $\phi_0 \in E$  such that

$$\phi(f) = (df)(\phi) = T^{-1}(d_E f)(\phi) = (d_E f)(\phi_0) = \phi_0(f)$$

for all  $f \in S$ , hence  $\phi = \phi_0$ . So  $\mathcal{X}(M) = E$ . ■

It is natural to conjecture that the hypothesis  $\|d_E f\| = \|df\|$  for all  $f \in \text{Lip}(M)$  should imply the condition  $|d_E f| = |df|$  needed in the preceding theorem. However, even if  $M$  is rigidly differentiable in the sense of Definition 30, this implication is false; see Theorem 58.

### B. Locality

Since we are treating abelian  $W^*$ -modules as bimodules with identical left and right actions, our metric derivations have a local character that contrasts with the bimodule derivations considered in [78]. This is seen in the following results.

**LEMMA 27.** *Let  $\delta: \text{Lip}(M) \rightarrow E$  be a metric derivation and let  $A \subset M$ . If  $f, g \in \text{Lip}(M)$  satisfy  $f|_A = g|_A$  almost everywhere then  $(\delta f)|_A = (\delta g)|_A$  almost everywhere.*

*Proof.* The statement that  $(\delta f)|_A = (\delta g)|_A$  is to be interpreted as meaning that  $\chi_A \cdot \delta f = \chi_A \cdot \delta g$ . By considering  $f - g$ , it will suffice to show that  $f|_A = 0$  implies  $(\delta f)|_A = 0$ . Let  $I = \{f \in \text{Lip}(M): f|_A = 0\}$ . This is a weak\*-closed ideal of  $\text{Lip}(M)$ , so by [78, Theorems 3 and 9]  $I^2$  is weak\*-dense

in  $I$ . Thus, for any  $f \in I$  we can find a pair of nets  $(f_i), (g_i) \subset I$  such that  $f_i g_i \rightarrow f$  weak\*. But then

$$f_i \delta(g_i) + \delta(f_i) g_i = \delta(f_i g_i) \rightarrow \delta(f)$$

weak\* in  $E$ , and since  $f_i, g_i \in I$  we have  $\delta(f_i g_i)|_A = 0$  for all  $i$ , hence  $(\delta f)|_A = 0$ . ■

**LEMMA 28.** *Let  $M$  be a measurable metric space, let  $A \subset M$ , and let  $f \in \text{Lip}(A)$ . Then there exists  $g \in \text{Lip}(M)$  such that  $g|_A = f$  and  $L(g) = L(f)$ .*

*Proof.* Without loss of generality suppose  $\|f\|_L = 1$ . For any positive measure set  $B \subset A$ , define  $f_B \in \text{Lip}(M)$  as in the proof of Proposition 14 by

$$f_B = \sup_C (\rho(B, C) \wedge 2) \cdot \chi_C.$$

Then define  $g \in \text{Lip}(M)$  by

$$g = \sup_B (a_B - f_B)$$

where  $a_B$  is the infimum of the essential range of  $f|_B$ , and the supremum is taken in  $L^\infty(M)$  (equivalently, as a limit in the weak\*-topology of the unit ball of  $\text{Lip}(M)$ ). Then  $f \geq a_B - f_B$  for all  $B$ , so  $f \geq g|_A$ . So if  $f \neq g|_A$  then we must have  $f \geq g + \varepsilon$  on some positive measure set  $B \subset A$ , contradicting the definition of  $a_B$  and the fact that  $g \geq a_B - f_B$ . So  $g$  has the desired properties. ■

**THEOREM 29.** *Let  $M$  be a measurable metric space and  $A \subset M$  a positive measure subset. Then  $\mathcal{X}(A) = \chi_A \cdot \mathcal{X}(M)$ .*

*Proof.* The natural map from  $\mathcal{X}(A)$  into  $\chi_A \cdot \mathcal{X}(M)$  is isometric by Lemma 28. Conversely, if  $\phi \in \chi_A \cdot \mathcal{X}(M)$  and  $f \in \text{Lip}(A)$ , we can apply  $\phi$  to  $f$  by first extending  $f$  via Lemma 28; by Lemma 27  $\phi$  is insensitive to the extension. So  $\mathcal{X}(A) = \chi_A \cdot \mathcal{X}(M)$ . ■

We now consider a special condition on  $M$ . We say a metric space is “rigidly differentiable” if, roughly speaking, its metric is captured by one-sided derivations, as opposed to the two-sided derivations needed in general [78]. For geometric purposes rigidly differentiable spaces, or spaces which are locally bi-Lipschitz equivalent to rigidly differentiable spaces, seem the most relevant. The precise definition is as follows.

DEFINITION 30. The measurable metric space  $M$  is *rigidly differentiable* if  $\|df\| = L(f)$  for every  $f \in \text{Lip}(M)$ , where  $d: \text{Lip}(M) \rightarrow \Omega(M)$  is the exterior derivative. Since  $\|df\| \leq L(f)$  automatically, an equivalent condition is

$$L(f) = \sup \{ \|\phi f\| : \phi \in \mathcal{X}(M), \|\phi\| = 1 \}$$

for all  $f \in \text{Lip}(M)$ .

THEOREM 31. Let  $M = (M, \mu, \rho)$  be a complete rigidly differentiable metric space and assume that every ball in  $M$  with strictly positive radius has strictly positive measure. Then  $\rho$  is a path-metric.

*Proof.* Note that we are assuming  $\rho$  is a pointwise metric (justified, say, by Theorem 20). We use the term “path-metric” in the sense of [11]: for each  $x, y \in M$  the distance between  $x$  and  $y$  is the infimum of  $L(f)$  as  $f$  ranges over all maps from  $[0, 1]$  into  $M$  with  $f(0) = x$  and  $f(1) = y$ .

Let  $\varepsilon > 0$ . Define a new metric  $\rho_\varepsilon$  by setting

$$\rho_\varepsilon = \inf \{ \rho(x_0, x_1) + \rho(x_1, x_2) + \cdots + \rho(x_{n-1}, x_n) \},$$

where the infimum is taken over all finite sequences  $x_0, \dots, x_n$  such that  $x_0 = x$ ,  $x_n = y$ , and  $\rho(x_{i-1}, x_i) \leq \varepsilon$  ( $1 \leq i \leq n$ ). It is easy to check that  $\rho_\varepsilon$  is a metric on  $M$ .

Now fix  $x, y \in M$  and define  $f(z) = \rho_\varepsilon(x, z)$ . If  $\rho(z_1, z_2) \leq \varepsilon$  then

$$|f(z_1) - f(z_2)| = |\rho_\varepsilon(x, z_1) - \rho_\varepsilon(x, z_2)| \leq \rho_\varepsilon(z_1, z_2) = \rho(z_1, z_2),$$

so  $L(f|_A) \leq 1$  for any set  $A \subset M$  with diameter at most  $\varepsilon$ . Then we can find  $g \in \text{Lip}(M)$  such that  $L(g) \leq 1$  (hence  $\|dg\| \leq 1$ ) and  $g|_A = f|_A$  by Lemma 28. Thus  $(df)|_A = (dg)|_A$  by Lemma 27; as  $M$  can be covered by positive measure balls of diameter at most  $\varepsilon$ , we conclude that  $\|df\| \leq 1$ . This implies  $L(f) \leq 1$  by differentiability, so

$$\rho_\varepsilon(x, y) = |f(x) - f(y)| \leq \rho(x, y).$$

As the reverse inequality is automatic, we have  $\rho(x, y) = \rho_\varepsilon(x, y)$ , and equality for all  $\varepsilon$  plus completeness implies that  $\rho$  is a path-metric [34, Théorème 1.8]. ■

The converse of Theorem 31 is false; there exist path-metric spaces  $M$  for which  $\mathcal{X}(M) = 0$  (see Theorem 41).

An easy nontrivial example of a rigidly differentiable space in which every ball of finite radius has measure zero is  $\mathbf{R}^2$  with Lebesgue measure and metric

$$\rho((x, y), (x', y')) = \begin{cases} |x - x'| & \text{if } y = y' \\ \infty & \text{if } y \neq y'. \end{cases}$$

Clearly, this space will remain rigidly differentiable if its metric is modified on a single horizontal line, but this can be done in such a way that  $\rho$  is no longer a path-metric.

## 5. EXAMPLES

We now determine  $\mathcal{X}(M)$  for various spaces  $M$ . Theorem 29 will be used repeatedly; it implies that our analysis need only be done locally. After restriction to a manageable subset of  $M$ , the main tools are Theorems 26 and 16.

### A. Atomic Measures, Stone Spaces, and Snowflake Spaces

We begin with some examples of metric spaces which admit no metric derivations, and hence are zero-dimensional in the sense that  $\mathcal{X}(M) = 0$ . The first result shows that the measure really is an essential ingredient in the construction of  $\mathcal{X}(M)$  (cf. [78, Corollary 7]).

**PROPOSITION 32.** *Let  $M$  be a metric space equipped with an atomic measure. Then  $\mathcal{X}(M) = 0$ .*

*Proof.* Let  $\delta \in \mathcal{X}(M)$ . Also let  $x \in M$  and let  $I = \{f \in \text{Lip}(M) : f(x) = 0\}$ . Then  $I$  is a weak\*-closed ideal of  $\text{Lip}(M)$ , so we have  $\delta(I) \subset I$  just as in the proof of Lemma 27. Now letting 1 denote the function which is constantly 1, we have  $\delta(1) = 0$  since  $\delta(1 \cdot 1) = 1 \cdot \delta(1) + \delta(1) \cdot 1 = 2\delta(1)$ . Thus  $f - f(x) \cdot 1 \in I$  implies

$$\delta(f) = \delta(f - f(x) \cdot 1) \in I;$$

i.e.,  $(\delta f)(x) = 0$ . This holds for all  $x \in M$ , so  $\delta f = 0$ . This shows that  $\mathcal{X}(M) = 0$ . ■

As a technical note, we should point out that our restriction to  $\sigma$ -finite measure spaces severely restricts the scope of Proposition 32. However, any  $l^\infty(X)$  is obviously finitely decomposable, regardless of the cardinality of  $X$ , and Proposition 32 in fact remains true at that level of generality. (Indeed, there is no real difficulty in doing everything up to this point with finitely decomposable measures. But it seems preferable to work with the more familiar property of  $\sigma$ -finiteness, since the only apparent drawback in doing so is the exclusion of uninteresting examples like  $l^\infty(X)$ .)

The next result shows that metric spaces with a certain disconnectedness property have no nonzero metric derivations. The condition implies that  $M$

is totally disconnected, but not every totally disconnected space satisfies it; in fact, any totally disconnected subset of  $\mathbf{R}^n$  with positive measure will have nontrivial  $\mathcal{X}(M)$  (Theorem 38).

**PROPOSITION 33.** *Let  $M$  be a measurable metric space and suppose the simple Lipschitz functions are weak\*-dense in  $\text{Lip}(M)$ . Then  $\mathcal{X}(M) = 0$ .*

*Proof.* Let  $\delta \in \mathcal{X}(M)$  and let  $f = \sum a_n \chi_{A_n} \in \text{Lip}(M)$  be a simple function with the sets  $A_n$  disjoint. Then  $f = a_n \cdot 1$  on  $A_n$ , so  $\delta(f)|_{A_n} = 0$  by Lemma 27; as this is true for all  $n$ , we have  $\delta(f) = 0$ . Density of the simple functions then implies that  $\delta = 0$ . ■

**COROLLARY 34.** *Let  $M$  be a measurable metric space which is uniformly discrete in the sense that there exists  $\varepsilon > 0$  such that every  $A, B \subset M$  satisfy  $\rho(A, B) = 0$  or  $\rho(A, B) \geq \varepsilon$ . Then  $\mathcal{X}(M) = 0$ .*

*Proof.* The uniform discreteness condition implies that  $\text{Lip}(M) \cong L^\infty(M)$ . So the simple functions are actually norm-dense in  $\text{Lip}(M)$ . ■

**COROLLARY 35.** *Let  $K$  be the middle-thirds Cantor set, equipped with any  $\sigma$ -finite Borel measure and with metric inherited from  $\mathbf{R}$ . Then  $\mathcal{X}(K) = 0$ .*

*Proof.* The simple functions are weak\*-dense in  $\text{Lip}(K)$  by Theorem 16 with  $k = 3$ . ■

Finally, we note that any “snowflake space,” in the terminology of [65], has trivial  $\mathcal{X}(M)$ .

**THEOREM 36.** *Let  $M = (M, \rho)$  be a compact metric space equipped with any  $\sigma$ -finite Borel measure and let  $0 < \alpha < 1$ . Let  $M^\alpha = (M, \rho^\alpha)$  be the corresponding “snowflake” or “Hölder” space. Then  $\mathcal{X}(M^\alpha) = 0$ .*

*Proof.* Let  $f \in \text{Lip}(M)$  and let  $L_\alpha(f)$  be its Lipschitz number with respect to the metric  $\rho^\alpha$ . Let  $\delta: \text{Lip}(M^\alpha) \rightarrow L^\infty(M^\alpha)$  be a metric derivation.

For any  $\varepsilon > 0$  there is a finite covering of  $M$  by sets  $A_1, \dots, A_n$  of  $\rho$ -diameter at most  $\varepsilon$ . Then  $L_\alpha(f|_{A_k}) \leq L(f) \cdot \varepsilon^{1-\alpha}$  for  $1 \leq k \leq n$  because exponentiating the metric expands the distance between any pair of points in any  $A_k$  by a factor of at least  $\varepsilon^{\alpha-1}$ . It follows from Lemmas 27 and 28 that

$$\|\delta(f)|_{A_k}\| \leq \|\delta\| \cdot L_\alpha(f|_{A_k}) \leq \varepsilon^{1-\alpha} \|\delta\| \cdot L(f)$$

for all  $k$ , and taking  $\varepsilon \rightarrow 0$  then yields  $\delta(f) = 0$ .



We have shown that  $\delta$  vanishes on  $\text{Lip}(M) \subset \text{Lip}(M^\alpha)$ . By Theorem 16 and [1, Lemma 3.3],  $\text{Lip}(M)$  is weak\*-dense in  $\text{Lip}(M^\alpha)$ , so weak\* continuity implies  $\delta = 0$ . Thus  $\mathcal{X}(M^\alpha) = 0$ . ■

### B. Lipschitz Manifolds

Next we consider the case of Lipschitz manifolds [21, 51, 60]. Note that this class includes all  $C^1$ -Riemannian manifolds.

**THEOREM 37.** *Let  $M$  be a Lipschitz manifold, equipped with Lebesgue measure class. Then  $\mathcal{X}(M)$ ,  $\Omega(M)$ , and  $d$  have their usual meanings (as in [31]).*

*Proof.* We check that every bounded measurable vector field (in the usual sense) gives rise to a metric derivation, and vice versa. By Theorem 29 it suffices to consider the case that  $M$  is a region in  $\mathbf{R}^n$ .

First consider the vector field  $\partial/\partial x_k$ . Every Lipschitz function is almost everywhere differentiable in any coordinate direction, with derivative in  $L^\infty(M)$  [26, Theorem 3.1.6]; and if  $f_i \rightarrow f$  boundedly weak\* in  $\text{Lip}(M)$  then  $(\partial f_i / \partial x_k)$  is bounded and

$$\int_M \left( \frac{\partial f_i}{\partial x_k} \right) g = - \int_M f_i \frac{\partial g}{\partial x_k} \rightarrow - \int_M f \frac{\partial g}{\partial x_k} = \int_M \left( \frac{\partial f}{\partial x_k} \right) g$$

for every  $g \in C^\infty(M)$  supported on the interior of  $M$ , which shows that  $\partial f_i / \partial x_k \rightarrow \partial f / \partial x_k$  weak\* in  $L^\infty(M)$ . So  $\delta(f) = \partial f / \partial x_k$  is a metric derivation.

Since the metric derivations constitute an  $L^\infty(M)$ -module, it follows that the vector field  $\sum_1^n f_k (\partial / \partial x_k)$  gives rise to a metric derivation for any  $f_1, \dots, f_n \in L^\infty(M)$ . So every bounded measurable vector field gives rise to a metric derivation.

Conversely, any metric derivation  $\delta$  is determined by its values  $f_k = \delta(x_k)$  on the coordinate functions since these functions generate  $\text{Lip}(M)$  by Theorem 16. So there are no metric derivations besides those which arise from bounded measurable vector fields. ■

Note that by Theorem 10 we can recognize  $\mathcal{X}(M)$  as the module of bounded measurable sections of a field of  $n$ -dimensional Banach spaces over  $M$ . This means that the tangent space at almost every point of  $M$  carries a natural Banach space norm, so that  $M$  can be said to have a Finsler structure. This recovers the main construction of [21]. But this phenomenon is not special to Lipschitz manifolds: all of the other examples we discuss, including the infinite-dimensional ones, have the same property.

### C. Rectifiable Sets

We consider  $(H^m, m)$  rectifiable and  $H^m$  measurable subsets of  $\mathbf{R}^n$  in the sense of [26, Paragraph 3.2.14]. Here  $H^m$  is  $m$ -dimensional Hausdorff measure.

**THEOREM 38.** *Let  $M$  be an  $(H^m, m)$  rectifiable and  $H^m$  measurable subset of  $\mathbf{R}^n$ . Then  $\mathcal{X}(M)$  is naturally identified with the module of bounded measurable sections of approximate tangent spaces [26, Paragraphs 3.2.16 and 3.2.19].*

*Proof.* By [26, Lemma 3.2.18]  $M$  can be decomposed into a countable union of sets each of which is bi-Lipschitz equivalent to a positive measure subset of  $\mathbf{R}^m$ . So by Theorem 29 we can reduce to the case that  $M$  is a positive measure subset of  $\mathbf{R}^m$ . In this case the approximate tangent space is  $\mathbf{R}^m$  at almost every point [26, Theorem 3.2.19], and the identification of metric derivations with bounded measurable vector fields then follows from Theorem 37 and Theorem 29. ■

### D. Sub-Riemannian Metrics

Let  $M$  be an  $n$ -dimensional Riemannian manifold and let  $B$  be a smooth  $k$ -dimensional subbundle of the tangent bundle  $TM$ . Define the length of any smooth path  $\gamma: [0, 1] \rightarrow M$  to be  $l(\gamma) = \int_0^1 \|\gamma'\| dt$ , and define a metric  $\rho$  on  $M$  by setting

$$\rho(x, y) = \inf\{l(\gamma) : \gamma(0) = x, \gamma(1) = y, \text{ and } \gamma'(t) \in B \text{ for all } t \in [0, 1]\}.$$

Let  $\rho'$  denote the usual Riemannian metric, defined by the same infimum taken over all smooth paths from  $x$  to  $y$ . Equip  $M$  with the usual Lebesgue measure class.

**THEOREM 39.**  *$\mathcal{X}(M)$  is naturally identified with the bounded measurable sections of  $B$ .*

*Proof.* By Theorem 29 it is sufficient to consider a small neighborhood of any point  $x \in M$ . Fix  $x$  and let  $\alpha$  be a diffeomorphism between a neighborhood of  $x$  and an open set in  $\mathbf{R}^n$  such that  $\alpha(x) = 0$ . Let  $v_1, \dots, v_n \in T_x M$  be orthonormal vectors, each of which either belongs to or is orthogonal to  $B_x$ , and let  $X_1, \dots, X_n$  be smooth vector fields on  $M$  such that  $X_i(x) = v_i$ . By projecting onto  $B$  and  $B^\perp$  and then orthonormalizing, we may assume that in a neighborhood of  $x$  the  $X_i$  are orthonormal ( $1 \leq i \leq n$ ) and  $X_1, \dots, X_k$  span  $B$ . Then, by composing  $\alpha$  with an invertible linear transformation on  $\mathbf{R}^n$ , we may assume that the vectors  $\alpha_* v_i$  are orthonormal with respect to the Euclidean metric on  $\mathbf{R}^n$  at 0. On a small enough neighborhood of 0 the vector fields  $\alpha_* X_i$  are then nearly orthonormal with respect to the Euclidean metric, and using a partition of unity we

may (1) find a metric on  $\mathbf{R}^n$  which agrees with the Euclidean metric outside a neighborhood of 0 and makes  $\alpha$  isometric near 0 and (2) extend the  $\alpha_* X_i$  to linearly independent vector fields on all of  $\mathbf{R}^n$ . Finally, applying Gramm-Schmidt we may take the  $X_i$  to be orthonormal with respect to the metric just introduced. Thus, in what follows we will assume that  $M = \mathbf{R}^n$  with a metric which is Euclidean outside a bounded region, and that there are globally defined orthonormal vector fields  $X_1, \dots, X_n$  the first  $k$  of which span  $B$ . The reduction given in this paragraph is due to Feres [28].

Now for  $1 \leq i \leq n$  let  $T_t^i$  ( $t \geq 0$ ) be the flow generated by  $X_i$ , and define  $\alpha_t^i: L^\infty(M) \rightarrow L^\infty(M)$  by  $\alpha_t^i(f) = f \circ T_t^i$ . Then for each  $i$ ,  $(\alpha_t^i)$  is a strongly continuous one-parameter group of automorphisms of  $L^\infty(M)$ . Let  $\delta_i: L^\infty(M) \rightarrow L^\infty(M)$  be its infinitesimal generator, defined by  $\delta(f) = \lim_{t \rightarrow 0} (f - \alpha_t^i(f))/t$ , with domain consisting of all  $f \in L^\infty(M)$  for which the limit exists in the weak\* sense.

Suppose  $1 \leq i \leq k$ . If  $x \in M$  then  $\gamma(t) = T_t^i(x)$  is a path whose tangent vector lies in  $B$  and has norm equal to one everywhere. Thus  $\rho(x, T_t^i(x)) \leq |t|$  for all  $t$ . So for any  $f \in \text{Lip}(M)$  we have

$$|f(x) - \alpha_t^i(f)(x)| = |f(x) - f(T_t^i(x))| \leq |t|L(f);$$

this shows that  $\|f - \alpha_t^i(f)\| \leq |t|L(f)$ , and [12, Proposition 3.1.23] then implies that  $f$  belongs to the domain of  $\delta_i$ . In fact, the above inequality shows that  $\|\delta_i(f)\| \leq L(f)$ , so that  $\delta_i$  is nonexpansive when regarded as a map from  $\text{Lip}(M)$  to  $L^\infty(M)$ . Furthermore,  $\delta_i$  is a  $W^*$ -derivation by [12, Proposition 3.1.6], so its restriction to  $\text{Lip}(M)$  is a metric derivation. Now for any  $f_1, \dots, f_k \in L^\infty(M)$ , we have  $\sum_1^k f_i \delta_i \in \mathcal{X}(M)$ , so every bounded measurable section of  $B$  defines a metric derivation.

To prove the converse, we invoke Theorem 26. Let  $E$  be the set of metric derivations of the form  $\sum_1^k f_i \delta_i$  described above. It is isometrically isomorphic as an  $L^\infty(M)$ -normed module to  $L^\infty(M, \mathbf{R}^n)$  (giving  $\mathbf{R}^n$  the Euclidean norm) and hence is reflexive. For the other hypothesis, let  $f \in \text{Lip}(M)$ . Convolving  $f$  with a  $C^\infty$  approximate unit of  $L^1(\mathbf{R}^n)$  produces a sequence of smooth functions, bounded in Lipschitz norm, which converge to  $f$  weak\* in  $L^\infty(M)$ . This shows that the smooth functions are weak\*-dense in  $\text{Lip}(M)$ , so we may assume  $f$  is smooth.

In the notation of Theorem 26, we must show that  $|d_E f| \geq |df|$ . (The reverse inequality is automatic.) Since  $\|df|_A\| \leq L(f|_A)$  for any positive measure set  $A$ , it will suffice to find, for any  $x \in M$  and  $\varepsilon > 0$ , a neighborhood  $A$  of  $x$  such that  $|d_E f| \geq L(f|_A) - \varepsilon$  on  $A$ .

Let  $a = (\sum_1^n (\delta_i f(x))^2)^{1/2}$  and  $b = (\sum_1^k (\delta_i f(x))^2)^{1/2}$ . Since  $f$  is smooth, we may find  $r > 0$  such that  $b - \varepsilon/2 \leq |d_E f| \leq b + \varepsilon/2$  and  $|df| \leq a + \varepsilon$  on the  $\rho'$ -ball of radius  $r$  about  $x$ . (The bound on  $|df|$  is due to the fact that every

metric derivation of  $\text{Lip}(M, \rho)$  is also a metric derivation of  $\text{Lip}(M, \rho')$ , hence is a linear combination of partial derivatives by Theorem 37.) Let  $s = r(b + \varepsilon/2)/3(a + \varepsilon)$ , and let  $A$  be the  $\rho'$ -ball of radius  $s$  about  $x$ .

Let  $y, z \in A$ ; we must show that  $|f(y) - f(z)|/\rho(y, z) \leq b + \varepsilon/2$ . Now  $\rho'(y, z) \leq 2s$ , so

$$|f(y) - f(z)| \leq (a + \varepsilon) \rho'(y, z) \leq 2r(b + \varepsilon/2)/3.$$

This shows that the desired inequality holds if  $\rho(y, z) \geq 2r/3$ . Otherwise, let  $\gamma: [0, 1] \rightarrow M$  be a constant velocity path from  $y$  to  $z$  which is everywhere tangent to  $B$  and whose total length is exactly  $\rho(y, z) < 2r/3$ ; this exists by [11, Lemma 2.1.2]. Since  $\rho'(y, x) \leq s < r/3$  it follows that  $\gamma$  lies entirely within the  $\rho'$ -ball of radius  $r$  about  $x$ , so that  $|d_E f|(\gamma(t)) \leq b + \varepsilon/2$  for any  $t \in [0, 1]$ . Thus

$$|f(y) - f(z)| \leq l(\gamma) \sup_{t \in [0, 1]} |d_E f|(\gamma(t)) \leq \rho(y, z)(b + \varepsilon/2),$$

as desired. ■

Theorem 39 is no longer true if we allow the dimension of  $B$  to vary; this is illustrated by the space  $M_1$  treated in Theorem 58 (see the comment following that theorem).

### E. The Sierpinski Carpet and Laakso's Space

Now we consider two more sophisticated variations on the Cantor set (which was already treated in Section 5.A).

Let  $S$  be the Sierpinski carpet obtained from the unit square by iterating the process of removing the middle ninth sub-square. That is,  $S$  is the set of points  $x = (x_1, x_2) \in [0, 1]^2$  such that for no  $n \in \mathbb{N}$  and  $1 \leq k, l \leq 3^{n-1}$  is it the case that

$$\frac{3k-2}{3^n} < x_1 < \frac{3k+1}{3^n} \quad \text{and} \quad \frac{3l-2}{3^n} < x_2 < \frac{3l+1}{3^n}.$$

We give  $S$  normalized Hausdorff measure  $\mu$ ; this means that if

$$S_{k,l,n} = S \cap \left( \frac{k-1}{3^n}, \frac{k}{3^n} \right) \times \left( \frac{l-1}{3^n}, \frac{l}{3^n} \right)$$

is nonempty then  $\mu(S_{k,l,n}) = 8^{-n}$ .

LEMMA 40. *Let  $f \in L^\infty(S)$  and let  $a$  belong to the essential range of  $f$ . For any  $\varepsilon_1, \varepsilon_2 > 0$  there exist  $k, l, n$  so that*

$$\mu(S_{k,l,n} \cap f^{-1}([a - \varepsilon_1, a + \varepsilon_1])) \geq 8^{-n}(1 - \varepsilon_2).$$

*Proof.* The usual proof of the Lebesgue differentiation theorem can be adapted to show that for almost every  $x \in S$  we have

$$\lim 8^n \int_{S_{k,l,n}} |f(y) - f(x)| dy = 0,$$

for any sequence of squares  $S_{k,l,n}$  each of which contains  $x$  and for which  $n \rightarrow \infty$ . For instance, the argument in [29, Sect. 3.4] can be carried over verbatim, replacing  $\mathbf{R}^n$  with  $S$  and “open ball” with “the closure of some  $S_{k,l,n}$ .”

Since this holds for almost every  $x \in S$ , it must hold for some  $x_0 \in f^{-1}((a - \varepsilon_1, a + \varepsilon_1))$  by the definition of essential range. Choose  $\varepsilon$  such that

$$[f(x_0) - \varepsilon, f(x_0) + \varepsilon] \subset [a - \varepsilon_1, a + \varepsilon_1];$$

then find a square  $S_{k,l,n}$  which contains  $x_0$  such that

$$8^n \int_{S_{k,l,n}} |f(y) - f(x_0)| dy \leq \varepsilon \varepsilon_2.$$

It follows that

$$8^n \mu(S_{k,l,n} \cap f^{-1}([f(x_0) - \varepsilon, f(x_0) + \varepsilon])) \geq 1 - \varepsilon_2,$$

which is enough. ■

THEOREM 41.  $\mathcal{X}(S) = 0$ .

*Proof.* Let  $\delta \in \mathcal{X}(S)$ . It will be enough to show that  $\delta(f) = 0$  when  $f$  is either of the two coordinate functions,  $f(x, y) = x$  or  $f(x, y) = y$ , since these generate  $\text{Lip}(S)$  by Theorem 16. The arguments in the two cases are the same, so take the first case and suppose  $\delta(f) \neq 0$ . Since this implies  $(a\delta)(f) \neq 0$  for any nonzero constant  $a$ , we can suppose without loss of generality that  $\|\delta(f)\| = \text{ess sup } \delta(f) = 1$ .

Define a sequence of piecewise-linear functions  $f_m \in C(S)$  by letting  $f_m(0, y) = 0$  and requiring

$$\frac{\partial f_m}{\partial x}(x, y) = \begin{cases} 1 & \text{if } 3k/3^m < x < (3k+1)/3^m \\ -1 & \text{if } (3k+1)/3^m < x < (3k+2)/3^m \\ 0 & \text{if } (3k+2)/3^m < x < (3k+3)/3^m. \end{cases}$$

Then  $f_m \rightarrow 0$  weak\* in  $\text{Lip}(S)$ . Also  $f_m - f$  is constant on the left  $3/8$  of each  $S_{k,l,m}$ ;  $f_m + f$  is constant on the middle  $2/8$  of each  $S_{k,l,m}$ ; and  $f_m$  is zero on the right  $3/8$  of each  $S_{k,l,m}$ .

Choose  $\varepsilon_1, \varepsilon_2 > 0$  such that  $c = 1/8 - 3\varepsilon_1/8 - 2\varepsilon_2 > 0$ , and apply the lemma to find a square  $S_{k,l,n}$  such that

$$\mu^{-1}(S_{k,l,n} \cap (\delta f)^{-1}([1 - \varepsilon_1, 1 + \varepsilon_1])) \geq 8^{-n}(1 - \varepsilon_2).$$

Now let  $m \geq n$  and consider  $8^n \int_{S_{k,l,n}} \delta(f_m)$ . This integral is zero on the  $3/8$  of  $S_{k,l,n}$  where  $f_m$  is zero. On the  $2/8$  where  $f_m + f$  is constant, we have  $|\delta(f_m)| = |\delta(f)| \leq 1$  and so the integral is not below  $-2/8$ . Of the remainder,  $\delta(f_m) \geq 1 - \varepsilon_1$  on a set of measure at least  $8^{-n}(3/8 - \varepsilon_2)$  and  $|\delta(f_m)| = |\delta(f)| \leq 1$  elsewhere, so the integral here is at least  $(3/8 - \varepsilon_2)(1 - \varepsilon_1) - \varepsilon_2$ . All together, we have

$$\int_{S_{k,l,n}} \delta(f_m) \geq 8^{-n}((3/8 - \varepsilon_2)(1 - \varepsilon_1) - \varepsilon_2 - 2/8) > c/8^n.$$

This holds for every  $m \geq n$ , so  $\int \delta(f_m) \chi_{S_{k,l,n}}$  does not go to zero as  $m \rightarrow \infty$ , contradicting weak\*-continuity of  $\delta$ . This shows that the assumption  $\delta(f) \neq 0$  is impossible. So  $\delta$  vanishes on the coordinate functions, hence  $\delta = 0$ . ■

A similar argument shows that the Sierpinski gasket (obtained by iterating the process of removing the middle fourth subtriangle from an equilateral triangle) supports no nonzero metric derivations. I have not tried to systematically extend the reasoning in Theorem 41 to other fractals, but it seems likely that the same sort of argument would apply to many fractal shapes with non-integral Hausdorff dimension.

The Sierpinski carpet is the closure of a sequence of finite graphs  $G_n$ . Namely,  $G_1$  is the boundary of the unit square  $[0, 1]^2$  and  $G_{n+1} = \bigcup_{k=1}^8 (G_n + v_k)/3$ , where  $v_1, \dots, v_8$  are the vectors  $(0, 0)$ ,  $(1, 0)$ ,  $(2, 0)$ ,  $(0, 1)$ ,  $(2, 1)$ ,  $(0, 2)$ ,  $(1, 2)$ ,  $(2, 2)$ . It may be worth noting that a reasonable one-dimensional differentiable structure on  $S$  can be obtained by setting aside Hausdorff measure and instead assigning zero measure to  $S - \bigcup G_n$  and using one-dimensional Lebesgue measure on each  $G_n$ .

In [47] Laakso constructed for every  $Q > 1$  a metric space which admits a weak Poincaré inequality and has Hausdorff dimension  $Q$ . These examples are something like a product spaces, where one factor is a unit cube and the other is a Cantor set. The Cantor set direction turns out to be invisible to our construction. To illustrate we consider the simplest version of Laakso's construction, although the argument would work equally well in the general case.

Let  $F_0 = [0, 1] \times K$  be the cartesian product of the unit interval and the middle-thirds Cantor set. For any  $n \geq 0$  and  $0 \leq i \leq 3^n$ , with  $i$  not divisible by 3, set equivalent every pair of points of the form

$$(i/3^n, y) \quad \text{and} \quad (i/3^n, y + 2/3^{n+1}),$$

whenever both belong to  $F_0$ . The quotient space  $F$  is then path-connected, and we give it the natural path metric. This is Laakso's space.

**THEOREM 42.**  $\mathcal{X}(F) \cong L^\infty(F)$ . For any  $g \in L^\infty(F)$  the map  $\delta_g: f \mapsto g \cdot \partial f / \partial x$  is a metric derivation from  $\text{Lip}(F)$  to  $L^\infty(F)$ , and every element of  $\mathcal{X}(F)$  is of this form.

*Proof.* The first statement follows from the second. Let  $g \in L^\infty(F)$  and let  $\pi: F_0 \rightarrow F$  be the natural projection. Then the map  $\delta: \text{Lip}(F_0) \rightarrow L^\infty(F_0)$  defined by  $\delta(f) = (g \circ \pi) \cdot \partial f / \partial x$  is a metric derivation (by the easy direction of Theorem 25, for example). Let  $T: L^\infty(F) \rightarrow L^\infty(F_0)$  be composition with  $\pi$ , and identify  $L^\infty(F)$  with  $T(L^\infty(F)) \subset L^\infty(F_0)$ ; then we have  $\delta_g(f) = \delta(f \circ \pi)$ , and it is a metric derivation because  $\delta$  is a metric derivation.

Now let  $\delta$  be any metric derivation from  $\text{Lip}(F)$  into  $L^\infty(F)$ . We want to show that it is of the above form. By taking  $g = \delta(x)$  (where  $x$  denotes the function  $(x, y) \mapsto x$ ) and replacing  $\delta$  with  $\delta - \delta_g$ , we may assume that  $\delta(x) = 0$ . We must show that  $\delta = 0$ .

Let  $\mathcal{A} = \{f \in \text{Lip}(F) : \delta(f) = 0\}$ . This is a weak\*-closed, unital subalgebra of  $\text{Lip}(F)$ , and it contains the function  $(x, y) \mapsto x$ , hence it contains every Lipschitz function which depends only on the first variable. By Lemma 27, it therefore contains any function which locally depends only on the first variable. This description is met by the following sequence of functions. Let  $n \geq 0$ . Then for any  $x \in [i/3^n, (i+1)/3^n]$  define

$$f_n(x, y) = \begin{cases} \min(x - i/3^n, (i+1)/3^n - x) & \text{if } y \in [3j/3^{n+1}, (3j+1)/3^{n+1}] \\ \max(i/3^n - x, (x - i + 1)/3^n) & \text{if } y \in [(3j-1)/3^{n+1}, 3j/3^{n+1}] \end{cases}$$

for any  $0 \leq j \leq 3^n$ . These functions all have Lipschitz number equal to 1 and belong to  $\mathcal{A}$  by the argument given above, and together with the function  $f(x, y) = x$  they separate any pair of points in  $F$  by at least half of their distance. Therefore, by Theorem 16 we conclude  $\mathcal{A} = \text{Lip}(F)$ , and we are done. ■

## F. Cheeger's Construction

By a pleasant coincidence, I sent Jeff Cheeger a copy of this paper at the same time that he sent me his paper [14] in which he also constructs a first-order derivative on metric measure spaces. While he requires some

conditions on the underlying space—a doubling condition and a Poincaré inequality—his techniques produce more detailed conclusions. On the whole Cheeger's methods are more “hard” analysis as opposed to the “soft” approach taken in this paper.

Using the machinery built up in [14] and in this paper it is fairly easy to show that the two constructions agree. This was worked out in consultation with Dr. Cheeger, and although I include it here the result should be credited to us jointly.

**THEOREM 43.** *Let  $M$  be a metric measure space which satisfies a doubling condition and a weak Poincaré inequality of type  $(1, p)$ . Then the exterior derivative constructed in [14] agrees with the one constructed here.*

*Proof.* For reasons explained in [14], without loss of generality we may assume that  $M$  is a length space. Cheeger's cotangent module consists of the bounded measurable sections of a finite-dimensional vector bundle, so it is trivially an abelian  $W^*$ -module. Also, Cheeger's exterior derivative is easily seen to be a derivation and it is weak\*-continuous by an argument involving the reflexivity of  $H_{1,p}(M)$  [15]. Thus it is a metric derivation and therefore is a quotient of our exterior derivative by Theorem 23. The adjoint of this quotient map is an embedding of Cheeger's tangent module into our  $\mathcal{X}(M)$ ; let  $E$  be the image of this embedding. Then for any  $f \in \text{Lip}(M)$  we have, in the notation of [14],

$$|d_E f| = g_f = \text{Lip } f \geq |df| \geq |d_E f|$$

almost everywhere, where the crucial inequality  $g_f = \text{Lip } f$  is [14, Theorem 6.1]. So the hypotheses of Theorem 26 are satisfied and thus Cheeger's tangent module is isomorphic to ours. This completes the proof. ■

The doubling condition is a version of finite-dimensionality; it fails in the infinite-dimensional spaces that we consider below in paragraphs G to I. The Poincaré inequality requires the existence of rectifiable curves and therefore also excludes rectifiable sets. However, both conditions hold for Lipschitz manifolds, sub-Riemannian metrics, Sierpinski spaces, and Laakso spaces, as well as the very recently constructed spaces in [10, 36], which we do not treat here. This should give an indication of the scope of the hypotheses in Theorem 43.

#### G. Hilbert cubes

Fix  $1 < p < \infty$  and a sequence  $(a_n) \in l^p(\mathbf{N})$  with  $a_n > 0$  for all  $n$ . Let  $M_p$  be the cartesian product  $M_p = \prod [0, a_n]$  with metric inherited from  $l^p(\mathbf{N})$ . Also, give  $M_p$  the product of normalized Lebesgue measure on each factor. Let  $q$  be the conjugate exponent to  $p$ .



We say that a sequence  $(f_n) \in L^\infty(M_p)$  is *weakly  $p$ -summable* if the partial sums  $\sum_1^N |f_n|^p$  are uniformly bounded in  $L^\infty(M_p)$ . We then define  $|(f_n)|^p = \sum |f_n|^p$  to be the supremum in  $L^\infty(M_p)$  of the partial sums, and denote the  $L^\infty(M_p)$ -normed module of all weakly  $p$ -summable sequences by  $l^p(L^\infty(M_p))$ .

LEMMA 44. *For any  $f \in \text{Lip}(M_p)$ , the sequence  $(\partial f / \partial x_n)$  is weakly  $q$ -summable.*

*Proof.* Note that  $\partial f / \partial x_n$  exists almost everywhere on  $M$ , by Rademacher's theorem [26, Theorem 3.1.6] plus Fubini's theorem. We will show that  $\sum_1^N |\partial f / \partial x_n|^q \leq L(f)$  almost everywhere, for any  $N \in \mathbb{N}$ . By Fubini's theorem it suffices to show this for Lipschitz functions on  $M_p^N = \prod_1^N [0, a_n]$ .

Let  $f \in \text{Lip}(M_p^N)$  and suppose  $f$  is differentiable at the point  $x = (x_1, \dots, x_N) \in M_p^N$ . Then for any  $b = (b_1, \dots, b_N) \in \mathbf{R}^N$  and  $\varepsilon > 0$  there exists  $r > 0$  such that

$$|f(x + rb) - f(x)| \geq (1 - \varepsilon) r \left| \sum b_n (\partial f / \partial x_n)(x) \right|.$$

But also

$$|f(x + rb) - f(x)| \leq rL(f) \sum |b_n|^p,$$

so taking  $\varepsilon \rightarrow 0$  we have

$$\left| \sum b_n (\partial f / \partial x_n)(x) \right| \leq L(f) \sum |b_n|^p.$$

As this holds for any  $b \in \mathbf{R}^N$ , we conclude that  $\sum_1^N |\partial f / \partial x_n(x)|^q \leq L(f)$  at every point  $x$  where  $f$  is differentiable. Since any Lipschitz function is differentiable almost everywhere, we are done. ■

LEMMA 45. *Let  $(f_n) \in l^p(L^\infty(M_p))$  and let  $\varepsilon_1, \varepsilon_2 > 0$ . Then there exists  $N \in \mathbb{N}$  and a subset  $A_N \subset M_p$  such that  $\mu(M_p - A_N) \leq \varepsilon_1$  and  $\sum_{N+1}^\infty |f_n|^p \leq \varepsilon_2$  almost everywhere on  $A_N$ .*

*Proof.* Fix a Borel representative of each  $f_n$ . For each  $N$  let

$$A_N = \left\{ x \in M_p : \sum_{N+1}^\infty |f_n(x)|^p \leq \varepsilon_2 \right\}.$$

Then the  $A_N$  are nested and  $\bigcup A_N = M_p$ ; since  $M_p$  has finite measure this implies that  $\mu(M_p - A_N) \leq \varepsilon_1$  for some  $N$ . ■

LEMMA 46.  $l^p(L^\infty(M_p))' \cong l^q(L^\infty(M_p))$ .

*Proof.* The product of a sequence in  $l^p(L^\infty(M_p))$  and a sequence in  $l^q(L^\infty(M_p))$  is weakly 1-summable, hence converges almost everywhere, hence is bounded and converges weak\* in  $L^\infty(M_p)$ . This shows that  $l^q(L^\infty(M_p))$  naturally embeds in  $l^p(L^\infty(M_p))'$ , and the embedding is clearly isometric.

Let  $\Phi \in l^p(L^\infty(M_p))'$  and for each  $m \in \mathbb{N}$  let  $g_m = \Phi(\delta_{m,n} \cdot 1_{M_p})$ , where  $\delta_{m,n} \cdot 1_{M_p}$  is the sequence whose  $m$ th term is the constant function  $1_{M_p} \in L^\infty(M_p)$  and whose other terms are all zero. We claim that  $(g_m) \in l^q(L^\infty(M_p))$ . If not, then for every  $C > 0$  there exist  $N \in \mathbb{N}$  and a positive measure set  $A \subset M_p$  such that  $\sum_1^N |g_m|^q \geq C$  on  $A$ . By shrinking  $A$  we may assume that each  $g_m$  ( $1 \leq m \leq N$ ) varies by less than  $N^{-1/q}$  on  $A$ . For  $1 \leq m \leq N$  let  $a_m$  be a constant such that  $|g_m - a_m| \leq N^{-1/q}$  on  $A$ , so that  $\sum_1^N |a_m|^q \geq C - 1$ . Then find a sequence  $(b_m)$  with  $l^p$ -norm one such that  $\sum_1^N a_m b_m \geq C - 1$ ; let  $f_m = b_m \cdot 1_{M_p}$ ; and observe that  $f = (f_m) \in l^p(L^\infty(M_p))$  and  $|(f_m)|_p = 1_{M_p}$  but

$$\Phi(f) = \sum_1^N f_m g_m \geq C - 2$$

on  $A$ . This contradicts boundedness of  $\Phi$  and establishes that  $(g_m) \in l^q(L^\infty(M_p))$ .

Finally, we must show that  $\Phi$  is given by summation against  $(g_m)$ . This is clearly true on any  $p$ -summable sequence which has only finitely many nonzero terms. Now suppose  $\Phi$  annihilates every such sequence. Then for any  $f = (f_n) \in l^p(L^\infty(M_p))$  and any  $\varepsilon_1, \varepsilon_2 > 0$  we can apply Lemma 45 to get a set  $A_N$  of measure at least  $1 - \varepsilon_1$  such that  $(\chi_{A_N} f_n)$  is within  $\varepsilon_2$  in norm from a sequence with only finitely many nonzero terms. Thus  $|\Phi(f)| \leq \varepsilon_2 \|\Phi\|$  on  $A_N$ . Taking  $\varepsilon_1, \varepsilon_2 \rightarrow 0$  shows that  $\Phi(f) = 0$ . This shows that every element of  $l^p(L^\infty(M_p))'$  is determined by its values on finite sequences, which completes the proof. ■

THEOREM 47.  $\mathcal{X}(M_p) \cong l^p(L^\infty(M_p))$ .

*Proof.* Let  $g = (g_n) \in l^p(L^\infty(M_p))$ . For any  $f \in \text{Lip}(M_p)$  the series

$$\delta_g(f) = \sum g_n (\partial f / \partial x_n)$$

converges weak\* in  $L^\infty(M_p)$  by Lemmas 44 and 46.

We claim that  $\delta_g$  is a metric derivation. Linearity and the derivation identity are easy, as is the inequality  $\|\delta_g\| \leq \|g\|$ . To check weak\*-continuity, suppose  $f_i \rightarrow f$  weak\* in the unit ball of  $\text{Lip}(M_p)$ . By taking a subnet, we may assume that  $\delta_g(f_i)$  converges weak\* to some  $h \in L^\infty(M_p)$ ; we must show that  $h = \delta_g(f)$ . Given  $\varepsilon_1, \varepsilon_2 > 0$  find a set  $A_N$  as in Lemma 45 for the sequence  $(g_n)$ . Let  $g_n^N = g_n$  if  $n \leq N$  and 0 otherwise. Then  $\delta_{g^N}$  is weak\*-continuous since it is a finite linear combination of partial derivatives, each of which is weak\*-continuous by the Theorem 37 plus Fubini's theorem. Also

$$\delta_g(f_i) - \delta_g(f) = (\delta_g(f_i) - \delta_{g^N}(f_i)) + \delta_{g^N}(f_i - f) + (\delta_{g^N}(f) - \delta_g(f)),$$

and on  $A_N$  the first and third terms are each at most  $\varepsilon_2$  in absolute value. Taking the limit, weak\*-continuity of  $\delta_{g^N}$  implies that  $|h - \delta_g(f)| \leq 2\varepsilon_2$  on  $A_N$ . Then taking  $\varepsilon_1, \varepsilon_2 \rightarrow 0$  establishes that  $h = \delta_g(f)$ , and we conclude that  $\delta_g$  is a metric derivation. Thus every  $p$ -summable sequence in  $L^\infty(M_p)$  gives rise to an element of  $\mathcal{X}(M_p)$ .

We invoke Theorem 26 to prove the converse. Let  $E$  be the set of all metric derivations which arise from  $p$ -summable sequences. It is reflexive by Lemma 46. For the other hypothesis, consider the Lipschitz functions on  $M_p$  of the form  $f \circ \pi^N$  for some  $N \in \mathbb{N}$  and  $f \in \text{Lip}(M_p^N)$ , where  $M_p^N = \prod_{i=1}^N [0, a_n]$  and  $\pi^N: M_p \rightarrow M_p^N$  is the natural projection. By Theorem 16 these functions are weak\*-dense in  $\text{Lip}(M_p)$ , so it will suffice to verify that  $|d(f \circ \pi^N)| = |d_E(f \circ \pi^N)|$  for  $f \in \text{Lip}(M_p^N)$ .

Let  $d^N$  be the exterior derivative on  $M_p^N$ . By Theorem 37 we know that the coordinate partial derivatives generate  $\mathcal{X}(M_p^N)$ , so

$$|d_E(f \circ \pi^N)| (z) = |d^N f| (\pi^N(z))$$

for almost all  $z \in M_p$ . Now let  $\delta \in \mathcal{X}(M_p)$ ,  $\|\delta\| \leq 1$ . For any norm-one  $L^\infty(M_p^N)$ -module map  $\Phi: L^\infty(M_p) \rightarrow L^\infty(M_p^N)$ , the map  $f \mapsto \Phi(\delta(f \circ \pi^N))$  is a metric derivation on  $M_p^N$ , hence

$$|\Phi(\delta(f \circ \pi^N))| \leq |d^N f|$$

on  $M_p^N$ . As this is true for all  $\Phi$ , Corollary 6 implies that

$$|\delta(f \circ \pi^N)| (z) \leq |d^N f| (\pi^N(z))$$

almost everywhere; so we conclude that

$$|d(f \circ \pi^N)| (z) \leq |d^N f| (\pi^N(z)) = |d_E(f \circ \pi^N)| (z)$$

for almost all  $z \in M_p$ . As the reverse inequality is automatic, this verifies the second hypothesis of Theorem 26 and therefore completes the proof. ■

The cases  $p = 1$  and  $p = \infty$  are less transparent, but they are still of interest because they allow us to falsify some natural conjectures. Thus, fix  $(a_n) \in l^1(\mathbb{N})$  with  $a_n > 0$  for all  $n$  and let  $M_1 = \prod [0, a_n]$ . Also let  $(b_n) \in c_0(\mathbb{N})$  with  $b_n > 0$  for all  $n$  and let  $M_0 = \prod [0, b_n]$ . Give  $M_1$  the  $l^1$  metric and  $M_0$  the  $c_0$  metric, and endow both with the product of normalized Lebesgue measure on each factor.

**PROPOSITION 48.** (a)  $\mathcal{X}(M_0)$  is not weak\*-closed in  $B(\text{Lip}(M_0), L^\infty(M_0))$  (the space of bounded linear maps from  $\text{Lip}(M_0)$  into  $L^\infty(M_0)$ ).

(b) There is a metric derivation  $\delta \in \mathcal{X}(M_1)$  which is not a  $W^*$ -derivation.

*Proof.* (a) For each  $n \in \mathbb{N}$  define  $\delta_n \in \mathcal{X}(M_0)$  by

$$\delta_n(f) = \sum_{k=1}^n (\partial f / \partial x_k),$$

and let  $\delta$  be a weak\*-cluster point of the (bounded) sequence  $(\delta_n)$  in the dual space  $B(\text{Lip}(M_0), L^\infty(M_0))$ .

Let  $f_m \in \text{Lip}(M_0)$  be the  $m$ th coordinate function,  $f_m(x) = x_m$ . Then  $f_m \rightarrow 0$  uniformly, hence weak\* in  $\text{Lip}(M_0)$ . However  $\delta(f_m) = \lim \delta_n(f_m) = 1_{M_0}$  for all  $m$ , so  $\delta(f_m)$  does not converge to zero weak\*, and hence  $\delta$  cannot be a metric derivation.

(b) Define  $\delta: \text{Lip}(M_1) \rightarrow L^\infty(M_1)$  by

$$\delta(f) = \sum_{k=1}^{\infty} a_k (\partial f / \partial x_k).$$

Then  $\delta \in \mathcal{X}(M_1)$  because  $\mathcal{X}(M_1)$  is a Banach space and the sequence  $(a_n)$  is summable. For each  $n \in \mathbb{N}$  define  $f_n(x) = x_n / a_n$ . Then  $f_n \rightarrow (1/2) \cdot 1_{M_1}$  weak\* in  $L^\infty(M_1)$ , while  $\delta(f_n) = 1_{M_1}$  for all  $n$ . Since  $\delta((1/2) \cdot 1_{M_1}) = 0$ , this shows that  $\delta$  is not a  $W^*$ -derivation. ■

## H. Banach Manifolds

For our purposes a Banach manifold is a metric space  $M$  which is locally bi-Lipschitz equivalent to the unit ball of some fixed Banach space. We appeal to Theorem 29 to reduce to the case where  $M = F$  is itself a Banach space. This ignores the issue that there is in general no canonical choice of measure or measure class on (the unit ball of) an infinite-dimensional

Banach space, so measure-theoretic complications may arise when one tries to build a manifold by patching local neighborhoods together.

Let  $F$  be a separable reflexive Banach space equipped with a  $\sigma$ -finite Borel measure  $\mu$ . We assume that there is a dense subspace  $F_0 \subset F$  with the property that  $\mu$  and its translation  $\mu_v$  by  $v$  are mutually absolutely continuous for any  $v \in F_0$ .

**LEMMA 49.** *Let  $X$  be a  $\sigma$ -finite measure space. Then  $L^\infty(X, F)$  is an  $L^\infty(X)$ -normed module and  $L^\infty(X, F)' \cong L^\infty(X, F')$ .*

*Proof.* By  $L^\infty(X, F)$  we mean the space of bounded measurable functions from  $X$  into  $F$ , modulo functions which vanish off of a null set. Verification of the  $L^\infty(X)$ -normed module property given in Theorem 2c is easy. For the second assertion, if  $\Phi \in L^\infty(X, F')$  and  $\phi \in L^\infty(X, F)$  then the map  $x \mapsto P(\Phi(x), \phi(x))$  is a bounded measurable function on  $X$ , where  $P: F' \times F \rightarrow \mathbf{R}$  is the natural pairing. From here it is straightforward to check that  $L^\infty(X, F')$  embeds isometrically in  $L^\infty(X, F)'$ . For instance, this can be done by showing that the simple functions in  $L^\infty(X, F')$  are norm-dense, and checking isometry on them.

Conversely, let  $\Phi \in L^\infty(X, F)'$ . Let  $S \subset F$  be a countable dense subset. For each  $v \in S$  fix a Borel version  $f_v$  of  $\Phi(v \cdot 1_X)$ . Then for any finite subset  $S_0$  of  $S$  the map

$$\sum_{v \in S_0} a_v v \mapsto \sum_{v \in S_0} a_v f_v(x)$$

is a bounded linear functional, of norm at most  $\|\Phi\|$ , on the linear span of  $S_0$ , for almost every  $x \in X$ . Thus there is a set  $X_0 \subset X$  of full measure such that for all  $x \in X_0$  the map  $\sum a_i v_i \mapsto \sum a_i f_{v_i}(x)$  is a bounded linear functional on the unclosed span of  $S$ . So for each  $x \in X_0$  there exists a unique element  $\Psi(x) \in F'$  such that  $\Psi(x)(v) = f_v(x)$  for all  $v \in S$ , and  $\|\Psi(x)\| \leq \|\Phi\|$ . The map  $x \mapsto \Psi(x)$  is measurable since the measurable structure on  $F'$  is generated by the linear functionals given by  $v \in S$ . So  $\Psi \in L^\infty(X, F')$ , and regarding the latter as embedded in  $L^\infty(X, F)'$  we have  $\Psi(v \cdot 1_X) = \Phi(v \cdot 1_X)$  for all  $v \in S$ . But the elements  $v \cdot 1_X$  generate  $L^\infty(X, F)$  as a Banach  $L^\infty(X)$ -module, so this implies that  $\Phi = \Psi$ . We conclude that  $L^\infty(X, F)' = L^\infty(X, F')$ . ■

**THEOREM 50.**  $\mathcal{X}(F) \cong L^\infty(F, F)$ .

*Proof.* Fix  $v \in F_0$ , and for  $t \in \mathbf{R}$  let  $\alpha_{tv}: L^\infty(F) \rightarrow L^\infty(F)$  be translation by  $tv$ . Then  $(\alpha_{tv})$  is a strongly continuous one-parameter group of automorphisms of  $L^\infty(F)$ , so as in the proof of Theorem 39 its generator  $\delta_v$  is a metric derivation when restricted to  $\text{Lip}(F)$ .

For any  $f \in \text{Lip}(F)$ , we have  $\|f - \alpha_w(f)\| \leq L(f) \|tv\|$ , so  $\|\delta_v\| \leq \|v\|$ . Conversely, find  $\phi \in F'$  such that  $\|\phi\| = 1$  and  $\phi(v) = \|v\|$ , and let  $f = (\phi \wedge C) \vee (-C)$  for some  $C > 0$ . Then  $f \in \text{Lip}(F)$  and  $L(f) = 1$ , but  $(\delta_v f)(w) = \|v\|$  if  $-C < \phi(w) < C$ . Taking  $C \rightarrow \infty$ , we conclude that  $|\delta_v| = \|v\|$  almost everywhere.

Thus if  $A_1, \dots, A_n \subset F$  are disjoint and  $v_1, \dots, v_n \in F_0$ , then  $\sum \chi_{A_i} \delta_{v_i}$  is a metric derivation and

$$\left| \sum \chi_{A_i} \delta_{v_i} \right| = \|v_i\|$$

on  $A_i$ . Taking completions, this shows that  $L^\infty(F, F)$  naturally isometrically embeds in  $\mathcal{X}(F)$ .

For the converse, we apply Theorem 26 with  $E = L^\infty(F, F)$  regarded as a subset of  $\mathcal{X}(F)$ . Reflexivity follows from the lemma. To verify the second hypothesis of Theorem 26, by Theorem 16 it suffices to consider functions in  $\text{Lip}(F)$  of the form  $f = f_0(\Phi_1, \dots, \Phi_n)$  for  $f_0$  a bounded  $C^\infty$  function on  $\mathbf{R}^n$  and  $\Phi_1, \dots, \Phi_n \in F'$ . Projecting  $F$  onto its quotient by the intersection of the kernels of  $\Phi_1, \dots, \Phi_n$  and applying the reasoning in Theorem 47 shows that for such  $f$  we have  $|d_E f| = |df|$  as needed. ■

## I. Wiener Space

Consider the Wiener space of continuous functions  $f: [0, \infty) \rightarrow \mathbf{R}$  such that  $f(0) = 0$ . According to [9, Chap. 3] there is a measurable isomorphism between this space, equipped with Wiener measure, and the space  $\mathbf{R}^\mathbf{N}$  = the product of countably many copies of  $\mathbf{R}$ , giving each factor normalized Gaussian measure. The structure of the Ornstein–Uhlenbeck operator and the Gross–Sobolev derivative are more transparent in the  $\mathbf{R}^\mathbf{N}$  picture, so we work there. In fact, the techniques we have introduced in preceding sections suffice to compute  $\mathcal{X}(\mathbf{R}^\mathbf{N})$  with little further effort.

The metric on  $\mathbf{R}^\mathbf{N}$  is defined by  $\rho(a, b) = (\sum |a_n - b_n|^2)^{1/2}$ , where  $a = (a_n)$  and  $b = (b_n)$ . This metric has infinite distances, and any ball of finite radius has measure zero, just as in the example mentioned following Theorem 31. Let  $l^2(L^\infty(\mathbf{R}^\mathbf{N}))$  denote the set of weakly 2-summable sequences  $(f_n) \subset L^\infty(\mathbf{R}^\mathbf{N})$ .

**THEOREM 51.**  $\mathcal{X}(\mathbf{R}^\mathbf{N}) \cong l^2(L^\infty(\mathbf{R}^\mathbf{N}))$ . This is naturally identified with the space of bounded measurable sections of the Hilbert bundle  $\mathbf{R}^\mathbf{N} \times l^2(\mathbf{N})$  over  $\mathbf{R}^\mathbf{N}$ .

*Proof.* For any 2-summable sequence  $(f_n) \in l^2(L^\infty(\mathbf{R}^\mathbf{N}))$  the series  $\sum_0^\infty f_n \partial / \partial x_n$  defines a metric derivation. This is shown by an argument

similar to the one used in the proof of Theorem 47. In the language of [57],  $l^2(L^\infty(\mathbf{R}^N))$  is a self-dual Hilbert module over  $L^\infty(\mathbf{R}^N)$ . It is therefore reflexive, and the second hypothesis of Theorem 26 can be verified by the technique of projection onto finitely many factors also used in the proof of Theorem 47.

The realization of  $l^2(L^\infty(\mathbf{R}^N))$  as sections of  $\mathbf{R}^N \times l^2(\mathbf{N})$  is given by identifying the 2-summable sequence  $(f_n) \in l^2(L^\infty(\mathbf{R}^N))$  with the section  $x \mapsto (f_n(x)) \in l^2(\mathbf{N})$ . ■

Comparison with [9, 75] shows that our exterior derivative on  $\mathbf{R}^N$  is precisely the Gross–Sobolev derivative.

## 6. DIRICHLET SPACES

### A. Intrinsic Metrics

In this section we relate our construction to two themes in the study of Dirichlet spaces. The first is the intrinsic metric associated to any Dirichlet form. This has been used in several places [6, 7, 8, 49, 69–72], and its geometric aspect was specifically considered in [69]. The second theme is the existence of a first-order differential calculus associated to certain Dirichlet spaces; this was hinted at in [8, 50] and thoroughly treated in [62, 63].

For basic material on Dirichlet forms we refer the reader to the classic texts [30, 68] and the more recent books [9, 52].

The intrinsic metric is most elegantly treated in the setting described in the next definition. We will connect the structure described here with traditional Dirichlet forms in Theorem 60.

**DEFINITION 52.** Let  $X$  be a  $\sigma$ -finite measure space. An  $L^\infty$ -diffusion form is a map  $\Gamma: \mathcal{D} \times \mathcal{D} \rightarrow L^\infty(X)$  where  $\mathcal{D}$  is a weak\*-dense, unital subalgebra of  $L^\infty(X)$ , such that

(a)  $\Gamma$  is bilinear, symmetric, and positive (i.e.  $\Gamma(f, f) \geq 0$  for all  $f \in \mathcal{D}$ );

(b)  $\Gamma(fg, h) = f\Gamma(g, h) + g\Gamma(f, h)$  for all  $f, g, h \in \mathcal{D}$  (that is—by symmetry— $\Gamma$  is a derivation in either variable); and

(c)  $\Gamma$  is closed in the sense that if  $(f_i) \subset \mathcal{D}$ ,  $\|\Gamma(f_i, f_i)\|$  is uniformly bounded, and  $f_i \rightarrow f$  weak\* in  $L^\infty(X)$ , then  $f \in \mathcal{D}$  and  $\Gamma(f_i, g) \rightarrow \Gamma(f, g)$  weak\* in  $L^\infty(X)$  for all  $g \in \mathcal{D}$ .

For example, if  $M$  is a Riemannian manifold then  $\mathcal{D} = \text{Lip}(M)$  and  $\Gamma(f, g) = \nabla f \cdot \nabla g$  describes an  $L^\infty$ -diffusion form. (By Theorems 23 and 37

the standard exterior derivative  $d$  is a metric derivation; hence, identifying the tangent and cotangent bundles, so is the gradient  $\nabla$ . This implies the desired closure property.) In Theorem 59 we generalize this example to include all  $L^\infty$ -diffusion forms.

**LEMMA 53.**  $|I(f, g)|^2 \leq I(f, f) I(g, g)$  almost everywhere, for all  $f, g \in \mathcal{D}$ .

*Proof.* Suppose the inequality fails. Then without loss of generality there exists  $\varepsilon > 0$ , a positive measure set  $A \subset X$ , and scalars  $a, b, c \in \mathbf{R}$  such that  $a \geq (bc)^{1/2} + \varepsilon$  and

$$I(f, g) \geq a, \quad b \geq I(f, f), \quad c \geq I(g, g)$$

on  $A$ . Choose  $h \in L^1(A)$  with  $h \geq 0$  and  $\int h = 1$ . Then integrating  $I$  against  $h$  gives rise to a positive semidefinite bilinear form  $\langle \cdot, \cdot \rangle_h$  on  $\mathcal{D}$ , and we have

$$\langle f, f \rangle_h^{1/2} \langle g, g \rangle_h^{1/2} \leq (bc)^{1/2} \leq a - \varepsilon \leq \langle f, g \rangle_h - \varepsilon,$$

contradicting the Cauchy-Schwartz inequality for  $\langle \cdot, \cdot \rangle_h$ . Thus the desired inequality must hold almost everywhere on  $X$ . ■

**THEOREM 54.** Let  $I$  be an  $L^\infty$ -diffusion form. Then there is a measurable metric  $\rho$  on  $X$  such that  $M = (X, \rho)$  satisfies  $\mathcal{D} = \text{Lip}(M)$  and  $\|I(f, f)\|^{1/2} = L(f)$  for all  $f \in \mathcal{D}$ .  $M$  is rigidly differentiable in the sense of Definition 30.

*Proof.* For each  $g \in \mathcal{D}$  with  $\|I(g, g)\| \leq 1$  let  $X_g$  be a copy of  $X$ ; then let  $Y = \bigcup X_g$  be their disjoint union.  $L^\infty(Y)$  is an abelian  $W^*$ -module over  $L^\infty(X)$  via the diagonal embedding of  $L^\infty(X)$  into  $L^\infty(Y)$ . Define  $\delta: \mathcal{D} \rightarrow L^\infty(Y)$  by  $\delta(f) = I(f, g)$  on  $X_g$ . Then  $\delta$  is a  $W^*$ -derivation and so Theorem 19 implies the existence of a measurable metric  $\rho$  on  $X$  such that  $M = (X, \rho)$  satisfies  $\mathcal{D} = \text{Lip}(M)$  and  $L(f) = \|\delta f\|$  for all  $f \in \mathcal{D}$ .

By the lemma we have  $|\delta f| \leq I(f, f)^{1/2}$  almost everywhere on  $X$ . Conversely, taking  $g = f/\|I(f, f)\|^{1/2}$  we have

$$\|\delta f\| \geq \|I(f, f)\|/\|I(f, f)\|^{1/2} = \|I(f, f)\|^{1/2}.$$

So  $L(f) = \|\delta f\| = \|I(f, f)\|^{1/2}$ . ■

Concretely, the measurable metric identified in Theorem 54 is given by

$$\rho(A, B) = \sup \{ \rho_f(A, B) : f \in \mathcal{D}, \|I(f, f)\| \leq 1 \}.$$

Thus, it is essentially the intrinsic metric mentioned earlier, except that the latter is a pointwise metric and cannot be defined without regularity



assumptions sufficient to make every  $f \in \mathcal{D}$  well defined at each point of  $X$ . This can always be assured by altering the underlying space via Theorem 20.

### B. Tangent and Cotangent Bundles

We now present Sauvageot's construction of an exterior derivative in the setting of  $L^\infty$ -diffusion forms and compare it to our exterior derivative. The approach of [62] has been altered slightly here to more clearly display its connection with Kähler differentiation; see, e.g., [35, Sect. 20] or [38, Sect. II.8].

**DEFINITION 55.** Let  $\Gamma$  be an  $L^\infty$ -diffusion form on  $L^\infty(X)$ . Let  $E_0$  be the algebraic tensor product  $E_0 = \mathcal{D} \otimes \mathcal{D}$ , equipped with the  $L^\infty(X)$ -valued inner product

$$\langle f_1 \otimes g_1, f_2 \otimes g_2 \rangle_{L^\infty} = f_1 f_2 \Gamma(g_1, g_2)$$

and regarded as a bimodule over  $\mathcal{D}$  with left and right actions given by  $f \cdot (g \otimes h) \cdot k = fg \otimes hk$ .

Let  $I = \{\phi \in E_0 : \langle \phi, \phi \rangle_{L^\infty} = 0\}$  and define  $E_1$  to be the sub-bimodule of  $E_0/I$  generated by the elements of the form  $f \otimes 1 - 1 \otimes f$  for  $f \in \mathcal{D}$ . A short calculation using the derivation identity (Definition 52b) shows that  $E_1$  is a monomodule, i.e.  $f\phi = \phi f$  for all  $f \in \mathcal{D}$  and  $\phi \in E_1$ .

Finally, let  $E$  be the set of bounded  $\mathcal{D}$ -module homomorphisms from  $E_1$  to  $L^\infty(X)$ . Note that  $E_1$  naturally embeds in  $E$  by identifying  $\phi \in E_1$  with the homomorphism  $\psi \mapsto \langle \psi, \phi \rangle_{L^\infty}$ . Furthermore,  $E$  is an  $L^\infty(X)$ -module with action  $f \cdot \Phi(\phi) = f\Phi(\phi)$  ( $f \in L^\infty(X)$ ,  $\Phi \in E$ ,  $\phi \in E_1$ ) and is a self-dual Hilbert module by [57, Theorem 4.2].

**DEFINITION 56.** Retain the notation used in Definition 55. Define  $\delta: \mathcal{D} \rightarrow E$  by

$$\delta(f) = 1 \otimes f - f \otimes 1.$$

It is easy to check that  $\delta$  is a derivation.

**PROPOSITION 57.** Let  $\Gamma$  be an  $L^\infty$ -diffusion form and let  $M = (X, \rho)$ ,  $E$ , and  $\delta$  be the associated structures identified in Theorem 54 and Definitions 55 and 56. Then  $\Gamma(f, g) = \langle \delta f, \delta g \rangle_{L^\infty}$  for all  $f, g \in \mathcal{D}$ ,  $\delta$  is a metric derivation, and  $\|\delta f\| = L(f)$  for all  $f \in \mathcal{D} = \text{Lip}(M)$ . There is a nonexpansive module homomorphism  $T: E \rightarrow \mathcal{X}(M)$  such that  $\delta = T^* \circ d$ .

*Proof.* Since  $\Gamma(1, f) = 0$  for any  $f \in \mathcal{D}$  by the derivation property, we have

$$\langle \delta f, \delta g \rangle_{L^\infty} = \langle 1 \otimes f, 1 \otimes g \rangle_{L^\infty} = \Gamma(f, g)$$

for all  $f, g \in \mathcal{D}$ . It follows from Theorem 54 that  $\|\delta f\| = \|\Gamma(f, f)\|^{1/2} = L(f)$  for all  $f \in \mathcal{D}$ . To see that  $\delta$  is a metric derivation, suppose  $f_i \rightarrow f$  boundedly weak\* in  $\text{Lip}(M)$  and let  $g, h \in \text{Lip}(M)$ . Then

$$\langle \delta f_i, g \otimes h \rangle_{L^\infty} = g\Gamma(f_i, h) \rightarrow g\Gamma(f, h) = \langle \delta f, g \otimes h \rangle_{L^\infty}$$

weak\* in  $L^\infty(X)$ , and this is sufficient to show that  $\delta f_i \rightarrow \delta f$  weak\* in  $E$  [57, Remark 3.9]. So  $\delta$  is a metric derivation.

$T: E \rightarrow \mathcal{X}(M)$  is defined by  $(T\phi)(f) = \langle \delta f, \phi \rangle_{L^\infty}$  for  $\phi \in E$  and  $f \in \text{Lip}(M)$ . For any  $f \in \text{Lip}(M)$  we then have

$$(T^* \circ d)(f)(\phi) = df(T\phi) = (T\phi)(f) = \langle \delta f, \phi \rangle_{L^\infty},$$

so that  $(T^* \circ d)(f) = \delta f$ . ■

In most natural examples the map  $T$  in Proposition 57 is an isometric isomorphism, so that  $\mathcal{X}(M) \cong \Omega(M)$  is the module of bounded measurable sections of a bundle of Hilbert spaces (see Corollary 24 and the comment following Theorem 10). However, this is not always the case. We now give two examples to illustrate what other behavior can occur. (Besides these examples, it is also instructive to consider abstract Wiener spaces, as discussed, e.g., in [9]. There one has a Hilbert space  $H$ , a Banach space  $F$ , and maps  $H \rightarrow F$  and  $F' \rightarrow H$  which correspond to the maps  $T$  and  $T^*$  of Proposition 57 acting on the tangent and cotangent spaces at a single point.)

Let  $M_1 = [0, 1]^2$  be the unit square with Euclidean metric and Lebesgue measure. Let  $S = M_1 \cap \mathbf{Q}^2$  be the set of points with rational coordinates. For each  $s = (x, y) \in S^2$  let  $O_s$  be an open subset of  $M_1$  which contains the line segment joining  $x$  and  $y$ ; since  $S^2$  is countable we can arrange that the total measure of  $A = \bigcup_{s \in S^2} O_s$  is less than one (indeed, arbitrarily close to zero). Let  $h \in L^\infty(M_1)$  satisfy  $0 \leq h \leq 1$  and  $h|_A = 1$ , and define

$$\Gamma_1(f, g) = h \nabla f \cdot \nabla g$$

for  $f, g \in \text{Lip}(M_1)$ . Let  $E$  and  $\delta$  be the associated Hilbert module and metric derivation identified in Definitions 55 and 56, and let  $d: \text{Lip}(M_1) \rightarrow \Omega(M_1)$  be the exterior derivative.

Let  $M_2 = [0, 1]^2$  be the unit square with  $l^1$  metric, i.e.

$$\rho(x, y) = |x_1 - y_1| + |x_2 - y_2|$$

for  $x, y \in M_2$ . Let  $B_h = (\mathbf{R} \times \mathbf{Q}) \cap M_2$ ,  $B_v = (\mathbf{Q} \times \mathbf{R}) \cap M_2$ , and  $B = B_h \cup B_v$ . Then  $B$  is a countable union of horizontal and vertical line segments, so we can find a probability measure  $\mu$  on  $M_2$  which is supported on  $B$  and whose restriction to any line segment is a positive multiple of Lebesgue measure. For  $f, g \in \text{Lip}(M_2)$  define

$$\Gamma_2(f, g) = \begin{cases} \left( \frac{\partial f}{\partial x_1} \right) \left( \frac{\partial g}{\partial x_1} \right) & \text{on } B_h \\ \left( \frac{\partial f}{\partial x_2} \right) \left( \frac{\partial g}{\partial x_2} \right) & \text{on } B_v \\ 0 & \text{elsewhere.} \end{cases}$$

(Note that  $B_h \cap B_v$  has measure zero, so the definition of  $\Gamma_2$  is consistent.)

**THEOREM 58.**  *$\Gamma_1$  and  $\Gamma_2$  are  $L^\infty$ -diffusion forms and  $M_1$  and  $M_2$  are the associated metric spaces. For  $f \in L^\infty(M_1)$  and  $g \in \text{Lip}(M_1)$  we have  $|f dg| = f |\nabla g|$  regardless of  $h$  but  $|f \delta g| = fh^{1/2} |\nabla g|$ .*

*Proof.* In both cases the metric of Theorem 54 agrees with the given metric on  $[0, 1]^2 \cap \mathbf{Q}^2$ , hence  $L(f) = \|\Gamma(f, f)\|^{1/2}$  for all  $f \in \mathcal{D}$ . The rest of the proof is routine. ■

Thus, taking  $f \in L^\infty(M_1)$  supported on  $\{x : h(x) \leq 1/2\}$ , say, and  $g \in \text{Lip}(M_1)$  with  $|\nabla g| = 1$  almost everywhere, we see that for  $M_1$  the map  $T$  of Proposition 57 is not isometric; and if  $h$  is zero on a positive measure set  $T^*$  will have a nonzero kernel. Whereas  $\mathcal{X}(M_2)$  and  $\Omega(M_2)$  are not even Hilbert modules, so  $T$  certainly cannot be isometric in this case.

$M_2$  can be viewed as a (nonsmooth) sub-Riemannian metric, as can  $M_1$  in the case  $h = \chi_A$ .

The next result, an easy corollary of Proposition 57, characterizes  $L^\infty$ -diffusions in metric terms.

**THEOREM 59.** *Let  $M$  be a measurable metric space, let  $E$  be a self-dual Hilbert module over  $L^\infty(M)$ , and let  $\delta : M \rightarrow E$  be a metric derivation which satisfies  $\|\delta f\| = L(f)$  for all  $f \in \text{Lip}(M)$ . Then  $\mathcal{D} = \text{Lip}(M)$  and  $\Gamma(f, g) = \langle \delta f, \delta g \rangle_{L^\infty}$  define an  $L^\infty$ -diffusion form on  $L^\infty(M)$ . Conversely, every  $L^\infty$ -diffusion form arises in this manner.*

*Proof.* It is easy to check that  $\Gamma$  is an  $L^\infty$ -diffusion form. The closure property of Definition 52 (c) follows from the hypothesis  $\|\delta f\| = L(f)$  (so

that bounded  $L^\infty$ -weak\* convergence in  $\mathcal{D}$  is weak\*-convergence in  $\text{Lip}(M)$ ) plus the fact that  $\delta$  is weak\*-continuous.

The converse statement is Proposition 57. ■

### C. Dirichlet Forms

We adopt the conventions of [9]. Thus, if  $X$  is a  $\sigma$ -finite measure space then a *Dirichlet form* on  $L^2(X)$  is a positive, symmetric, closed bilinear map  $\mathcal{E}: D(\mathcal{E}) \times D(\mathcal{E}) \rightarrow \mathbf{R}$ , where  $D(\mathcal{E})$  is a dense subspace of  $L^2(X)$ , such that  $f \in D(\mathcal{E})$  implies  $f \wedge 1 \in D(\mathcal{E})$  and

$$\mathcal{E}(f \wedge 1, f \wedge 1) \leq \mathcal{E}(f, f).$$

Associated to any Dirichlet form  $\mathcal{E}$  there is a *sub-Markovian symmetric semi-group*  $(P_t^{(2)})$  ( $t \geq 0$ ), which is a norm continuous semigroup of self-adjoint contractions on  $L^2(X)$  that satisfy  $0 \leq P_t^{(2)} f \leq 1$  whenever  $0 \leq f \leq 1$ . The infinitesimal generator  $A^{(2)}$  of the semigroup corresponding to the Dirichlet form  $\mathcal{E}$  satisfies  $D((-A^{(2)})^{1/2}) = D(\mathcal{E})$  and  $\mathcal{E}(f, g) = -\langle A^{(2)} f, g \rangle$  for all  $f \in D(A^{(2)})$  and  $g \in D(\mathcal{E})$ .

The restrictions of the operators  $P_t^{(2)}$  to  $L^1(X) \cap L^2(X)$  extend to a norm continuous contraction semigroup  $P_t^{(1)}$  on  $L^1(X)$  whose infinitesimal generator is denoted  $A^{(1)}$ . The intersection of  $D(A^{(2)})$  with  $D(A^{(1)})$  is a core for both  $A^{(1)}$  and  $A^{(2)}$ , and on this intersection the two operators agree.

$\mathcal{E}$  is a *diffusion form* if its jump and killing parts are zero ([68], [30]). If the form is regular, this is equivalent to strong locality (follows from Theorem 11.10 of [68]; see also Section 4.1(i) of [69]).

The Dirichlet form  $\mathcal{E}$  is said to *admit a carré du champ* if  $D(A^{(1)}) \cap L^\infty(X)$  is an algebra. (See [9] for several equivalent conditions.) In this case there is a positive, symmetric, bilinear map  $\Gamma^{(2)}: D(\mathcal{E}) \times D(\mathcal{E}) \rightarrow L^1(X)$  which satisfies

$$\mathcal{E}(fh, g) + \mathcal{E}(gh, f) - \mathcal{E}(h, fg) = \int h \Gamma^{(2)}(f, g)$$

for all  $f, g, h \in D(\mathcal{E}) \cap L^\infty(X)$ . If  $\mathcal{E}$  is a diffusion form, this can be simplified to the equality

$$2\mathcal{E}(f, g) = \int \Gamma^{(2)}(f, g)$$

for all  $f, g \in D(\mathcal{E})$ .

We now show how to derive  $L^\infty$ -diffusion forms from diffusion forms which admit a *carré du champ*. Morally, one should be able to go in the reverse direction as well by defining  $\mathcal{E}(f, g) = \int \Gamma(f, g)/2$  when an  $L^\infty$ -diffusion form  $\Gamma$  is given. Two difficulties arise, however. First, there is the question of whether  $\Gamma(f, g)$  is integrable for sufficiently many  $f$  and  $g$  (a problem that can presumably be addressed by taking some care in choosing  $\mu$  from its measure class); more seriously, it seems that the closure

condition on  $\Gamma$  is weaker than the corresponding closure condition on  $\mathcal{E}$ . So there may be cases where one cannot pass in this direction, though I do not know of any examples.

**THEOREM 60.** *Let  $X$  be a  $\sigma$ -finite measure space and let  $\mathcal{E}$  be a diffusion form on  $L^2(X)$  which admits a carré du champ. Then there is a unique  $L^\infty$ -diffusion form  $\Gamma$  which agrees with  $\Gamma^{(2)}$  on a common core.*

*Proof.* Define a weak\*-continuous contraction semigroup  $P_t^{(\infty)}$  on  $L^\infty(X)$  by letting  $P_t^{(\infty)}$  be the adjoint of  $P_t^{(1)}$ . For any  $f, g \in L^1(X) \cap L^\infty(X)$  we have  $\int P_t^{(2)}(f) g = \int f P_t^{(2)}(g)$  since  $P_t^{(2)}$  is self-adjoint; and since  $P_t^{(1)}$  agrees with  $P_t^{(2)}$  on  $L^1(X) \cap L^\infty(X)$  this implies that  $P_t^{(\infty)} = P_t^{(1)}$  on  $L^1(X) \cap L^\infty(X)$ . Thus, the infinitesimal generator  $A^{(\infty)}$  of  $(P_t^{(\infty)})$  agrees with  $A^{(1)}$  on  $S = D(A^{(1)}) \cap L^\infty(X)$ .

For  $f, g \in S$  define

$$\Gamma(f, g) = -(A^{(\infty)}(fg) - fA^{(\infty)}(g) - gA^{(\infty)}(f)).$$

This makes sense because  $S$  is an algebra by [9, Theorem 4.2.1], and  $\Gamma = \Gamma^{(2)}$  on  $S$  by [9, Theorem 4.2.2]. The derivation property follows from the fact that  $\mathcal{E}$  is a diffusion form by [50, Sect. 1.5]; the remaining issue is closability of  $\Gamma$ .

For  $A, B \subset X$  define

$$\rho(A, B) = \sup \{ \rho_f(A, B) : f \in S \text{ and } \Gamma(f, f) \leq 1 \}.$$

By Theorem 16,  $S$  is weak\*-dense in  $\text{Lip}(M)$  where  $M = (X, \rho)$ . We must define  $\Gamma(f, g)$  for all  $f, g \in \text{Lip}(M)$ .

First, for  $f \in \text{Lip}(M)$  and  $g \in S$ , set

$$\Gamma(f, g) = \lim \Gamma(f_i, g)$$

where  $(f_i) \subset S$  and  $f_i \rightarrow f$  boundedly weak\* in  $\text{Lip}(M)$ . To see that this definition makes sense, we must show that  $f_i \rightarrow 0$  implies  $\Gamma(f_i, g) \rightarrow 0$ . Suppose  $f_i \rightarrow 0$  weak\* and let  $h \in S$ ; then

$$\begin{aligned} \int \Gamma(f_i, g) h &= \int -A^{(\infty)}(f_i g) h + f_i A^{(\infty)}(g) h + g A^{(\infty)}(f_i) h \\ &= \int -f_i g A^{(1)}(h) + f_i A^{(\infty)}(g) h + f_i A^{(1)}(gh) \\ &\rightarrow 0, \end{aligned}$$

as desired. Thus  $\Gamma(f, g)$  is well defined when  $f \in \text{Lip}(M)$  and  $g \in S$ .

To define  $\Gamma(f, g)$  for any  $f, g \in \text{Lip}(M)$  we use the key trick in the proof of [57, Theorem 3.2]. Fix  $f, g \in \text{Lip}(M)$ . For any  $h \in L^1(X)$ ,  $h \geq 0$ , let  $H_h$  be the pre-Hilbert space consisting of the set  $S$  with the pseudonorm  $\|k\|^2 = \int \Gamma(k, k) h$ . Then the maps  $\phi_f: k \mapsto \int \Gamma(f, k) h$  and  $\phi_g: k \mapsto \int \Gamma(g, k) h$  are bounded linear functionals on  $H_h$ , hence both are represented by elements of the Hilbert space completion of  $H_h$ . Define  $\Gamma(f, g)_h$  to be the inner product of  $\phi_f$  and  $\phi_g$ , and observe that  $|\Gamma(f, g)_h| \leq L(f) L(g) \|h\|_1$ . We can then define  $\Gamma(f, g)_h$  for all  $h \in L^1(X)$  by linearity and let  $\Gamma(f, g) \in L^\infty(X)$  be defined by

$$\int \Gamma(f, g) h = \Gamma(f, g)_h.$$

$\Gamma$  is now closed because  $f_i \rightarrow f$  boundedly weak\* in  $\text{Lip}(M)$  implies  $\phi_{f_i} \rightarrow \phi_f$  for all  $h \in S$ , hence  $\Gamma(f_i, g)_h \rightarrow \Gamma(f, g)_h$ . The other desired properties of  $\Gamma$  hold by continuity. ■

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